Explicit and combined estimators for parameters of stable distributions

Jacques Lévy Véhel[1] Anne Philippe[2] Caroline Robet [2]

[1] INRIA Rennes Bretagne Atlantique & Case Law Analytics,

[2] Université de Nantes, Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629

Abstract: This article focuses on the estimation of the stability index and scale parameter of stable random variables. While there is a sizable literature on this topic, no precise theoretical results seem available. We study an estimator based on log-moments, which always exist for such random variables. The main advantage of this estimator is that it has a simple closed form expression. This allows us to prove an almost sure convergence result as well as a central limit theorem. We show how to improve the accuracy of this estimator by combining it with previously defined ones. The closed form also enables us to consider the case of non identically distributed data, and we show that our results still hold provided deviations from stationarity are "small". Using a centro-symmetrization, we expand the previous estimators to skewed stable variables and we construct a test to check the skewness of the data. As applications, we show numerically that the stability index of multistable Lévy motion may be estimated accurately and consider a financial log, namely the S&P 500, where we find that the stability index evolves in time in a way that reflects with major financial events.

Keywords: averaging estimates; misspecifed model; moment estimate; Monte Carlo approximation; stable distribution

1 Introduction

The class of α-stable distributions is ubiquitous in probability: such distributions appear as the limit of normalized sums of independent and identically distributed random variables.

A random variable X is said to have α-stable distribution with $\alpha \in (0, 2]$ if for any $n \geq 2$ there is a real D_n such that $n^{1/\alpha}X + D_n$ has the same distribution as $X_1 + \cdots + X_n$, the sum of n independent copies of X (see Samorodnitsky and Taqqu (1994) for equivalent definitions and properties). This probability distribution admits a continuous probability density, that is not known in closed form, except for Gaussian distributions, Cauchy distributions, Lévy distributions and constants. Non-Gaussian stable distributions are a model of choice for real world phenomena exhibiting jumps. Indeed, for $\alpha < 2$, their density exhibit "heavy tails", resulting in a power-law decay of the probability of extreme events. They have been used extensively in recent years for modeling in domains such as biomedicine (see Salas-Gonzalez et al., 2013), geophysics (see Yang et al., 2009), economy and finance (see Mandelbrot, 1997), Internet traffic (see Dimitriadis et al., 2011) and more. A stable distribution is characterized by four parameters:

- a stability parameter, denoted $\alpha \in (0, 2]$. The value $\alpha = 2$ corresponds to Gaussian distribution. For non-Gaussian stable distribution $\alpha \in (0, 2)$, it governs the heaviness of the tail. Its density decreases as the power function $x^{-\alpha-1}$ when $|x|$ tends to infinity (see (6.1) in Appendix or Property 1.2.15 in Samorodnitsky and Taqqu (1994) for details).

- a scale parameter usually denoted σ, (that is proportional to variance in the Gaussian case),

- a location parameter μ similar to the mean in the case of Gaussian distributions,

- a skewness parameter β ranging in $[-1, 1]$.

Hereafter, we write $X \sim S_\alpha(\sigma, \beta, \mu)$ to indicate that X has a stable distribution. The characteristic function ϕ, is given by (see Samorodnitsky and Taqqu, 1994, for details):

$$\phi(t) = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha \left(1 - i\beta \text{sign}(t) \tan(\frac{\pi\alpha}{2})\right) + i\mu t\right), & \text{if } \alpha \neq 1, \\ \exp\left(-\sigma |t| \left(1 + i\beta \text{sign}(t) \frac{2 \log |t|}{\pi}\right) + i\mu t\right), & \text{if } \alpha = 1, \end{cases}$$

where

$$\text{sign}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

Our main aim in this work is to estimate these parameters. This is an important step to use stable distributions for real world phenomena modeling, which is not trivial. The maximum likelihood estimate (MLE), that is the natural estimate for a parametric problem, is difficult to use (except in few cases, for example Gaussian distribution, Cauchy distribution, ...). Bergström (1952) gives a series representation of the density function. From this representation, DuMouchel (1973) establishes the asymptotic theory of MLE. Under conditions ensuring the existence of MLE, he proves the consistency and the asymptotic normality. See also DuMouchel (1975) for results on the Fisher information. Despite this property of optimality, it is nevertheless difficult to calculate the MLE. In practice, finding the MLE requires numerical approximations. Different methods has been proposed for instance through Fourier inversion (see Nolan (2001)). However, such procedures entail approximation errors that cannot be easily assessed.

Another difficulty is that, except in the Gaussian case, stable random variables have infinite moments of order at least the stability index. More precisely, if X is a $S_\alpha(\sigma, \beta, \mu)$ stable random variable with $0 < \alpha < 2$, then we have $E[|X|^p] < \infty$ if and only if $0 < p < \alpha$ (see Samorodnitsky and Taqqu, 1994, Prop 1.2.16). This property implies that the non-Gaussian stable random variables do not possess a finite variance, and, in some cases, a well-defined mean. Therefore, a standard method of moments cannot be used.

A number of estimators are of common use, such as the ones proposed by Fama and Roll (1971), McCulloch (1986) and Koutrouvelis (1980, 1981). A difficulty with these estimators is that they do not possess a simple closed form expression. As a consequence, and to the best of our knowledge, no theoretical results are known about them, such as almost sure convergence and central limit theorems. Their asymptotic distributions as well as asymptotic variances are thus only accessible through numerical simulations. Another drawback of not having explicit and simple closed forms is that it is difficult to assess theoretically their performance in situations that slightly depart from the classical assumptions of identical and independent random samples. This is nevertheless desirable when one wishes to deal with real world data, which will often not verify these ideal hypotheses.

The parameter α can be also interpreted as a tail index. Indeed, the asymptotic tail behavior of the stable distribution is Pareto when $\alpha \neq 2$, i.e. it exists $C > 0$ such that $P(X > x) \sim Cx^{-\alpha}$ as x tends to infinity (see Samorodnitsky and Taqqu (1994)). Therefore, all the estimators of index of regularly varying distributions can be applied to α. See for instance Hill (1975); Hall (1982); De Haan L. (2006); Resnick (2007) for a review, and McCulloch (1997); DuMouchel (1983); Fofack and Nolan (1999) for applications to stable distributions.

The construction of estimators for parameters of stable distribution is also related to more recent studies on Lévy processes. An important challenge is to characterize the activity of jumps. In the more general context of semi-martingales, Aït-Sahalia and Jacod (2009); Jing et al. (2012) propose estimates for the jump activity index based on discrete high-frequency observations. For a Lévy process, this jump activity index corresponds to the Blumenthal–Getoor index, which can be estimated in different ways (see for instance Belomestny (2010) for spectral approach). In the particular case of stable Lévy process, this index is just the stable index α.

Recently, Falconer and Lévy Véhel (2018a,b) construct a new class of processes called self-stabilizing processes. The stability index at time t depends on the value of the process at time t. For a self-stabilizing process $(Z_t)_{t\in\mathbb{R}^+}$, its limit distribution after scaling around t, is an $\alpha(Z(t))$-stable process. The estimation of the function α is a difficult issue that requires an easy-to-calculate estimator with good properties for small samples.

Our main aim in this work is to investigate the theoretical properties of a generalized method of moments with log-moments. This idea is not new, as it has long been remarked that log-moments always exist for stable random variables and that it is convenient to work with them. Ma and Nikias (1995) consider the same estimator as the one we study in the symmetric case.

They apply it to blind channel identification while Wang et al. (2015) use this estimator for α-stable noise in a laser gyroscope's random error. Kuruoglu (2001) considers the general case with four parameters based on a symmetrization of the observations. In these articles, the asymptotic properties are not addressed. Owing to its simple expression, we are able to prove almost sure convergence and a central limit theorem both in an identically and independent framework and in a case of slight deviation from stationarity. We compare the performance of our estimator with the Koutrouvelis regression method (see Koutrouvelis, 1980, 1981). The results depend on the value of α and on the size of the sample. We then combine these two estimators using a technique recently developed in Lavancier and Rochet (2016) to enhance their performance, especially in the case of small samples. As applications, we show numerical experiments both on synthetic data (symmetric Lévy multistable motion) and on a financial log (S&P 500), which confirm our theoretical results that the estimator is able to track smooth enough variations of the stability index in time.

In Section 2, we study estimators of α and σ for symmetric (that is, when $\mu = \beta = 0$) stable random variables: log-moments estimators and a combined estimator build with the Koutrouvelis ones. In Section 3, we expand the log-moment, Koutrouvelis and combined estimators to the skewed case studying two ways for the adaptation of the log-moments estimator. The properties of log-moment estimators also allow us to propose a method for testing the skewness of the data. In Section 4, we investigate the case of non-identically distributed observations, we prove robustness of the log-moments estimators under some conditions for the perturbations. In Section 5, we perform numerical experiments involving multistable Lévy motion and real data with the study of a financial index.

2 Estimation methods

After the theoretical study of the log-moment estimate, we apply the procedure described in Lavancier and Rochet (2016) to provide a combined estimator for the parameters $\alpha \in (0, 2)$ and $\sigma \in \mathbb{R}_+^*$.

2.1 Symmetric case for log-moments

For a symmetric stable distribution closed form expressions are available for absolute log-moments (Le Guével, 2013), which allow one to derive expressions for estimating α and σ. First, note the following property:

Proposition 2.1. *Let* $Z \sim S_\alpha(1, 0, 0)$ *with* $0 < \alpha < 2$*. We have* $E[|\log |Z||^p] < \infty$ *for all* $p > 0$*.*

Proof. See Appendix. $\square$

These expectations may be computed explicitly by remarking that:

$$E[(\log |Z|)^p] = \left. \frac{d^p E[|Z|^t]}{dt^p} \right|_{t=0}, \tag{2.1}$$

and by using the following result:

Proposition 2.2. *Let $Z \sim S_\alpha(1,0,0)$ with $0 < \alpha < 2$. For all $0 < t < \min(\alpha, 1)$, we have*

$$E[|Z|^t] = \frac{\Gamma(1 - t/\alpha)}{\Gamma(1-t)\cos(\pi t/2)}. \tag{2.2}$$

Proof. See Appendix. □

We deduce that $E[\log |Z|] = \left(\frac{1}{\alpha} - 1\right)\gamma$ and $\mathrm{Var}(\log |Z|) = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12}$, where γ is the Euler constant.

Theorem 2.3. *Let $(X_1, \ldots, X_n)$ be a sequence of independent and identically distributed standard symmetric stable random variables $S_\alpha(1,0,0)$ with $0 < \alpha < 2$. Define*

$$\widehat{\alpha}_n(X_1, \ldots, X_n) = \frac{\gamma}{\gamma + \frac{1}{n}\sum_{i=1}^n \log |X_i|}.$$

Then, $\widehat{\alpha}_n \xrightarrow{a.s.} \alpha$ when $n \to +\infty$. Moreover, with $f(x) = \frac{\pi^2}{6x^2} + \frac{\pi^2}{12}$,

$$\frac{\sqrt{n}(\widehat{\alpha}_n - \alpha)\gamma}{\widehat{\alpha}_n^2 \sqrt{f(\widehat{\alpha}_n)}} \xrightarrow{d} \mathcal{N}(0,1). \tag{2.3}$$

Proof. The proof of this result may be found in Appendix. □

In general, σ is unknown and we must use a joint estimation of both parameters. Let W be a stable variable with parameter $S_\alpha(\sigma,0,0)$. By taking $Z = \frac{W}{\sigma}$, we have $Z \sim S_\alpha(1,0,0)$ and we deduce the log-moments of W by using the log-moments of Z. We get

$$E[\log |W|] = \left(\frac{1}{\alpha} - 1\right)\gamma + \log \sigma$$

and

$$\mathrm{Var}(\log |W|) = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12}. \tag{2.4}$$

Theorem 2.4. *Let $(X_1, \ldots, X_n)$ be a sequence of independent and identically distributed symmetric stable random variables $S_\alpha(\sigma,0,0)$ with $0 < \alpha < 2$ and $\sigma > 0$. Define the estimators*

$\widehat{\alpha}_{LOG}^{(n)} = \widehat{\alpha}_{LOG}^{(n)}(X_1, \ldots, X_n)$ *and* $\widehat{\sigma}_{LOG}^{(n)} = \widehat{\sigma}_{LOG}^{(n)}(X_1, \ldots, X_n)$ *by*

$$\begin{cases} \widehat{\alpha}_{LOG}^{(n)} = \left(\max \left(\frac{6}{\pi^2 n} \sum_{i=1}^{n} \left[\log|X_i| - \frac{1}{n} \sum_{k=1}^{n} \log|X_k| \right]^2 - \frac{1}{2}, \frac{1}{4} \right) \right)^{-1/2}, \\ \widehat{\sigma}_{LOG}^{(n)} = \exp\left(\frac{1}{n} \sum_{i=1}^{n} \log|X_i| - \left(\frac{1}{\widehat{\alpha}_{LOG}^{(n)}} - 1 \right) \gamma \right). \end{cases} \tag{2.5}$$

Then,

$$(\widehat{\sigma}_{LOG}^{(n)}, \widehat{\alpha}_{LOG}^{(n)}) \xrightarrow{a.s.} (\sigma, \alpha) \text{ when } n \to +\infty. \tag{2.6}$$

Moreover,

$$\sqrt{n} \left(\begin{pmatrix} \widehat{\sigma}_{LOG}^{(n)} \\ \widehat{\alpha}_{LOG}^{(n)} \end{pmatrix} - \begin{pmatrix} \sigma \\ \alpha \end{pmatrix} \right) \xrightarrow{d} \mathcal{N}(0, F_{\alpha,\sigma} G_{\alpha,\sigma} \Sigma_{\alpha,\sigma} \, G_{\alpha,\sigma}^{\top} \, F_{\alpha,\sigma}^{\top}), \tag{2.7}$$

where

$$F_{\alpha,\sigma} = \begin{pmatrix} \sigma & \gamma\sigma/\alpha^2 \\ 0 & 1 \end{pmatrix}, G_{\alpha,\sigma} = \begin{pmatrix} 1 & 0 \\ 6\left(\left(\frac{1}{\alpha} - 1\right)\gamma + \log\sigma\right)\alpha^3/\pi^2 & -3\alpha^3/\pi^2 \end{pmatrix}, \tag{2.8}$$

$$\Sigma_{\alpha,\sigma} = \begin{pmatrix} \mathrm{Var}(\log|X_1|) & \mathrm{Cov}(\log|X_1|, (\log|X_1|)^2) \\ \mathrm{Cov}(\log|X_1|, (\log|X_1|)^2) & \mathrm{Var}((\log|X_1|)^2) \end{pmatrix}. \tag{2.9}$$

Proof. The proof of this result may be found in the appendix. □

For the parameter of interest α, we have an explicit form of the limiting distribution and its variance can be consistently estimated.

Corollary 2.5. *Under the same assumption as Theorem 2.4,*

1. *the asymptotic variance in* (2.7) *for* $\widehat{\alpha}_{LOG}^{(n)}$ *only depends on* α *and is equal to*

$$\tau_\alpha^2 := \frac{36}{\pi^4} \frac{1}{|6\mu_2/\pi^2 - 1/2|^3} \left[\mu_4 - \mu_2^2\right]$$

where μ_2, μ_4 *are central moments (see* (2.4) *and* (6.2) *)*

$$\mu_2 = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12} \text{ and } \mu_4 = \pi^4 \left(\frac{3}{20\alpha^4} + \frac{1}{12\alpha^2} + \frac{19}{240} \right).$$

2. *Moreover, we have*

$$T_n = \frac{\sqrt{n}}{\tau_n} (\widehat{\alpha}_{LOG}^{(n)} - \alpha) \xrightarrow{d} \mathcal{N}(0, 1), \tag{2.10}$$

where $\tau_n^2 = \tau^2_{\widehat{\alpha}_{LOG}^{(n)}}$ *is the plug-in estimates of* τ_α^2.

Proof. This is an immediate consequence of Theorem 2.4. □

The interest of (2.10) is to provide asymptotic confidence intervals for the parameter α. It also possible to apply this result for testing null hypothesis of the form $\alpha \in A$ where A is a subset of $(0, 2)$.
For example, the choice $A = [1, 2)$ allows to test the existence of the first moment. The critical region of the test is of the form $\{\widehat{\alpha}_{LOG}^{(n)} < 1 + \frac{\tau_n}{\sqrt{n}} q_w\}$ where q_w is the standard Gaussian quantile of order w. This test has asymptotic significance level w and it is consistent under the alternative hypothesis "X_1 is not integrable".

2.2 Combined estimator

A way to improve the performance of the log-moment estimate, is to aggregate different estimates. We want to construct an estimator of α which will be at least as good as the best estimator, for each α, for small samples. We build a new estimator for the parameters α and σ using a combined estimator whose general procedure of construction is described in Lavancier and Rochet (2016). In our special case, $\theta = (\alpha, \sigma)^\top$ are the parameters to estimate and we have access to p estimators for α and q estimators for σ.

Let $\widehat{\alpha}_{(p)}$ (resp.$\widehat{\sigma}_{(q)}$) be the collection of p (resp. q) estimates of α (resp. σ). We consider averaging estimators of θ of the form

$$\widehat{\theta}_\lambda = \lambda^\top \begin{pmatrix} \widehat{\alpha}_{(p)} \\ \widehat{\sigma}_{(q)} \end{pmatrix}, \quad \lambda \in \Lambda, \tag{2.11}$$

where $\lambda^\top$ denotes the transpose of λ and $\Lambda \subset \mathbb{R}^{(p+q)\times 2}$ is a subset of $(p+q) \times 2$ matrices.

A convenient way to measure the performance of $\widehat{\theta}_\lambda$ is to compare it to $\widehat{\theta}^\star$, defined as the best linear combination $\widehat{\theta}_\lambda$ obtained for a non-random vector $\lambda \in \Lambda$. Specifically, $\widehat{\theta}^\star$ is the linear combination $\lambda^{\star\top} \left(\begin{smallmatrix} \widehat{\alpha}_{(p)} \\ \widehat{\sigma}_{(q)} \end{smallmatrix}\right)$ minimizing the mean squared error (MSE), i.e.

$$\lambda^\star = \underset{\lambda \in \Lambda}{\operatorname{argmin}}\, E[\| \widehat{\theta}_\lambda - \theta \|^2].$$

Clearly, the larger the set Λ is, the better it will be. However, choosing the whole space $\Lambda = \mathbb{R}^{(p+q)\times 2}$ is generally not exploitable. We must impose some conditions on the set Λ in order to have an explicit form for $\lambda^\star$.

Define $J = \left(\begin{smallmatrix} \mathbf{1}_p & \mathbf{0}_p \\ \mathbf{0}_q & \mathbf{1}_q \end{smallmatrix}\right)$ where $\mathbf{1}_k$ is the vector composed of k ones and where $\mathbf{0}_k$ is the vector composed of k zeros. We consider the maximal constraint set

$$\Lambda_{\max} = \{\lambda \in \mathbb{R}^{(p+q)\times 2} / \lambda^\top J = I_2\}$$

with I_2 the identity matrix. The mean squared error $E[\| \widehat{\theta}_\lambda - \theta \|^2]$ is minimized on the set $\Lambda_{\max}$

for a unique solution $\lambda^\star = \Sigma^{-1}J(J^\top\Sigma^{-1}J)^{-1}$, where Σ is the Gram matrix

$$\Sigma = E\left(\left(\begin{smallmatrix}\widehat{\alpha}_{(p)}-\alpha\\ \widehat{\sigma}_{(q)}-\sigma\end{smallmatrix}\right)\left(\begin{smallmatrix}\widehat{\alpha}_{(p)}-\alpha\\ \widehat{\sigma}_{(q)}-\sigma\end{smallmatrix}\right)^\top\right). \tag{2.12}$$

This result is proved in Lavancier and Rochet (2016) (page 178). Since the matrix Σ is unknown, the averaging estimator $\widehat{\theta}_{\max}$ is obtained by replacing Σ by its estimation $\widehat{\Sigma}$:

$$\widehat{\lambda}_{\max} = \widehat{\Sigma}^{-1}J(J^\top\widehat{\Sigma}^{-1}J)^{-1}, \quad \widehat{\theta}_{\max} = \widehat{\lambda}_{\max}^\top \begin{pmatrix} \widehat{\alpha}_{(p)} \\ \widehat{\sigma}_{(q)} \end{pmatrix}. \tag{2.13}$$

Different strategies are described in Lavancier and Rochet (2016) to estimate Σ depending on information available on combined estimates.

In Section 2.3, we combine the log-moment estimate with the well-known Koutrouvelis estimator. In the absence of an explicit or asymptotic form for the variance of the Koutrouvelis estimator, we estimate Σ using the parametric bootstrap.

2.3 Numerical performance of the individual and combined estimators

In this section, we provide a numerical comparison of the log-moment estimate and the Koutrouvelis estimate (see Koutrouvelis, 1980, 1981). The choice of this particular estimator is motivated by the results reported in Weron (1995) showing that it performs usually better than other methods such as the Fama-Roll and McCulloch ones.

Then, to improve the performances of the Koutrouvelis estimator and the log-moments estimator we implement the aggregation method introduced in Section 2.2.

Definition of Koutrouvelis estimate. The Koutrouvelis (1980, 1981) estimator is based on exploiting the explicit expression of the iterated logarithm of the characteristic function ϕ. In the symmetric case, it takes the particularly simple form

$$\log(\log(|\phi(t)|^2)) = \log(2\sigma^\alpha) + \alpha\log|t|. \tag{2.14}$$

The empirical characteristic function given by $\widehat{\phi}_n(t) = \frac{1}{n}\sum_{j=1}^n e^{itX_j}$ based on i.i.d observations (X_j) is a consistent estimator of ϕ. We estimate these parameters by regressing $y = \log(\log(|\widehat{\phi}_n(t)|^2))$ on $w = \log|t|$ in the model $y_k = m + \alpha w_k + \epsilon_k$ where $m = \log(2\sigma^\alpha)$, $t_k = \frac{\pi k}{25}$ for $k \in \{1, \ldots, K\}$ with K depending on the parameter α and on the sample size, and ϵ_k denotes an error term. In our simulations, we use an easier version of the Koutrouvelis regression method which is more adapted for the symmetric case (see Weron, 1995). We describe the algorithm for an observed sample of size n:

- **Stopping parameters.** Fix the admissible error tol and the maximum number of iterations $iter_{max}$ if the admissible error is not reached. In all simulations, we take $tol = 0.05$

and $iter_{max} = 10$.

- **Initialization.** A regression applied to the McCulloch (1986) quantile method provides initial estimates $\widehat{\alpha}$ and $\widehat{\sigma}$. We fix a first estimate of the scale parameter s for the scaled sample $\left(\frac{X_j}{\sigma}\right)_{j\in\{1,\dots,n\}}$ by the deterministic value $\widehat{s} = 2$.
- **Recursive loop.** While the number of iterations is less than $iter_{max}$ and $|\widehat{s} - 1| > tol$,
  - find the number K of points in the regression depending on $\widehat{\alpha}$ as in the classical Koutrouvelis regression,
  - define $w = (w_k)_{k\in\{1,\dots,K\}}$ and $y = (y_k)_{k\in\{1,\dots,K\}}$ by

$$w_k = \log |t_k| \quad \text{and} \quad y_k = \log\left(-\log\left(\left|\widehat{\phi}_n\left(\frac{t_k}{\widehat{\sigma}}\right)\right|^2\right)\right),$$

    where $t_k = \frac{\pi k}{25}$ for $k \in \{1, \dots, K\}$,
  - compute $\bar{w}$ and $\bar{y}$ the empirical mean of samples $(w_k)_{k\in\{1,\dots,K\}}$ and $(y_k)_{k\in\{1,\dots,K\}}$,
  - compute the new $\widehat{\alpha}$ given by

$$\widehat{\alpha} = \min\left(\frac{\sum_{k=1}^{K}(w_k - \bar{w})(y_k - \bar{y})}{\sum_{k=1}^{K}(w_k - \bar{w})^2}, 2\right),$$

  - set the new $\widehat{s}$ by $\widehat{s} = \exp\left(\frac{\bar{y} - \widehat{\alpha}\bar{w} - \log(2)}{\widehat{\alpha}}\right)$,
  - set the new $\widehat{\sigma}$ by $\widehat{\sigma} = \widehat{\sigma}\widehat{s}$.

This modified version of Koutrouvelis gives performances (in terms of mean squared errors) similar to the original. However, it is much faster because this version does not necessitate the estimation of the parameters β and μ, which requires the numerical inversion of matrices of size $n \times n$.

Remark 1. As already pointed out by Weron (1995), simulation studies show that the recursive scheme stabilizes very quickly. With a high probability, the algorithm stops before $iter_{max}$ iterations, and so the admissible error tol is reached. Moreover, we do not observe a significant effect on the estimates when we reduce the value of tol. To support these remarks, we realize the following numerical experiments. We simulate 10^5 independent copies of stable samples with parameters $(\alpha, \sigma, \beta, \mu) = (1.8, 1, 0, 0)$ and the sample size is $n = 500$. The Koutrouvelis algorithm is executed for two values of the admissible error $tol = 0.01$ and 0.05. For all replications, the algorithm stops before reaching the number of iterations $iter_{max}$. Table 1 gives the estimated probability distribution of the number of iterations. When the tol parameter decreases, the average computing time increases without any significant improvement of approximation quality. The mean squared error for the α estimate is 4.3×10^{-3} (with a precision of 10^{-4}) for both values of $tol = 0.01,\ 0.05$. Moreover, the L^2-norm of the difference between the estimations is of the order of 10^{-5}.

Nb of iterations	1	2	3	≥ 3
tol= 0.01	25%	74%	1%	0%
tol= 0.05	88%	12%	0%	0%

Table 1: Estimated probability distributions of the number of iterations in Koutrouvelis algorithm. Estimations are done on $r = 100000$ independent copies. The sample are simulated with the following parameters : $n = 500$, $\alpha = 1.8$, $\sigma = 1$ and $\beta = \mu = 0$.

Comparison of individual estimates. For each pair of values (α, σ), r independent samples of size n of independent stable random variables are generated. The empirical mean squared error of the sampling distribution of α and σ is given by

$$MSE_\alpha = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\alpha}_i - \alpha)^2, \qquad MSE_\sigma = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\sigma}_i - \sigma)^2,$$

where $\widehat{\alpha}$ (resp $\widehat{\sigma}$) is an estimator of α (resp σ).

In the sequel, we use the abbreviation "KOUT" and "LOG" to refer respectively to the Koutrouvelis and log-moment estimator. For each α, the behaviors of $\widehat{\alpha}_{KOUT}$ and $\widehat{\alpha}_{LOG}$ are similar for all values of σ (Table 2) whereas, for each value of σ, $\widehat{\sigma}_{KOUT}$ and $\widehat{\sigma}_{LOG}$ improve when α is increasing (Table 3). Besides, when α is fixed, $\widehat{\sigma}_{KOUT}$ and $\widehat{\sigma}_{LOG}$ have the same behavior for all σ.

With a simulation study, we compare the empirical mean squared errors of α and σ for the methods introduced earlier. Tables 2 and 3 show that the log-moment estimator performs better than the Koutrouvelis one when $\alpha < 1$, while the converse is true for $\alpha > 1$, with the differences in performance increasing for extreme values of α.

Combined estimator. The Koutrouvelis regression estimator and the log-moment estimator are complementary in the sense that the Koutrouvelis regression estimator is preferable when $\alpha > 1$ whereas the log-moment estimate becomes better when $\alpha < 1$. Applying the method described in Section 2.2 with these estimators, we hope to get an estimate which will be at least

		α =0.2	α =0.6	α =1	α =1.4	α =1.8
σ =10	LOG	$9.06\ 10^{-5}$	$9.06\ 10^{-4}$	$4.67\ 10^{-3}$	$1.97\ 10^{-2}$	$3.10\ 10^{-2}$
	KOUT	$4.70\ 10^{-4}$	$2.35\ 10^{-3}$	$3.75\ 10^{-3}$	$8.20\ 10^{-3}$	$4.17\ 10^{-3}$
σ =1	LOG	$8.07\ 10^{-5}$	$1.06\ 10^{-3}$	$4.47\ 10^{-3}$	$2.20\ 10^{-2}$	$3.19\ 10^{-2}$
	KOUT	$4.27\ 10^{-4}$	$2.11\ 10^{-3}$	$3.91\ 10^{-3}$	$7.58\ 10^{-3}$	$4.28\ 10^{-3}$
σ =0.1	LOG	$8.93\ 10^{-5}$	$9.59\ 10^{-4}$	$4.43\ 10^{-3}$	$2.20\ 10^{-2}$	$2.97\ 10^{-2}$
	KOUT	$4.66\ 10^{-4}$	$1.96\ 10^{-3}$	$4.22\ 10^{-3}$	$7.58\ 10^{-3}$	$4.29\ 10^{-3}$

Table 2: Mean squared error for $\widehat{\alpha}_{LOG}$ and $\widehat{\alpha}_{KOUT}$ ($r = 500$ and $n = 500$).

		α =0.2	α =0.6	α =1	α =1.4	α =1.8
σ =10	LOG	7.10	9.86 10^{-1}	6.08 10^{-1}	6.07 10^{-1}	5.43 10^{-1}
	KOUT	11.5	9.92 10^{-1}	4.82 10^{-1}	3.56 10^{-1}	1.72 10^{-1}
σ =1	LOG	8.20 10^{-2}	9.45 10^{-3}	6.52 10^{-3}	6.98 10^{-3}	5.77 10^{-3}
	KOUT	1.18 10^{-1}	9.82 10^{-3}	4.44 10^{-3}	3.83 10^{-3}	1.56 10^{-3}
σ =0.1	LOG	6.73 10^{-4}	8.77 10^{-5}	6.71 10^{-5}	6.54 10^{-5}	5.89 10^{-5}
	KOUT	8.63 10^{-4}	8.60 10^{-5}	4.75 10^{-5}	3.35 10^{-5}	1.82 10^{-5}

Table 3: Mean squared error for $\widehat{\sigma}_{LOG}$ and $\widehat{\sigma}_{KOUT}$ ($r = 500$ and $n = 500$).

as good as the best estimator, for each α. The estimate $\widehat{\sigma}_{KOUT}$ is better than $\widehat{\sigma}_{LOG}$ except for the small values of α where $\widehat{\sigma}_{LOG}$ slightly outperforms. Therefore we use only $\widehat{\sigma}_{KOUT}$ in the combination. We will see later that, for technical reasons, it is not relevant to include too many estimators in the combination.

We consider combined estimate of the form

$$\widehat{\lambda}^\top \begin{pmatrix} \widehat{\alpha}_{KOUT} \\ \widehat{\alpha}_{LOG} \\ \widehat{\sigma}_{KOUT} \end{pmatrix}$$

where

$$\widehat{\lambda} = \widehat{\Sigma}^{-1} J (J^\top \widehat{\Sigma}^{-1} J)^{-1}, \quad J = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and where $\hat{\Sigma}$ is the parametric bootstrap estimate of the Gram matrix. Note that the parametric approaches proposed by Lavancier and Rochet (2016) cannot be applied in our context since an explicit or asymptotic form for the variance of the Koutrouvelis estimator is unknown. The bootstrap procedure is the following. We compute a first estimation of the parameters by

$$\widehat{\alpha}_0 = \frac{\widehat{\alpha}_{KOUT} + \widehat{\alpha}_{LOG}}{2}, \quad \widehat{\sigma}_0 = \widehat{\sigma}_{KOUT}.$$

We simulate B samples of size n of a symmetric stable distribution with parameters $\widehat{\alpha}_0$ and $\widehat{\sigma}_0$. Then, the three estimators are computed, which gives $\widehat{\alpha}^{(b)}_{KOUT}$, $\widehat{\alpha}^{(b)}_{LOG}$ and $\widehat{\sigma}^{(b)}_{KOUT}$ for $b = 1, \ldots, B$, and the matrix Σ is estimated by the empirical covariance matrix of sample $\left(\widehat{\alpha}^{(b)}_{KOUT}, \widehat{\alpha}^{(b)}_{LOG}, \widehat{\sigma}^{(b)}_{KOUT}\right)_{b=1,\ldots,B}$.

The consistency of parameter estimates justifies the correct use of this approach (see Theorem 2.4 and Koutrouvelis (1980)). Moreover, by construction, $\hat{\Sigma}$ is positive definite matrix and so the inversion of $\hat{\Sigma}$ is always possible.

Experiments show that errors entailed by the estimation of Σ are negligible compared to the advantage of having several estimators for small samples. Note that similar estimators could be built by combining more than 2 estimators for α. For example, it would be possible to add the McCulloch quantile estimator. We can also add $\widehat{\sigma}_{LOG}$ for σ. However, this would increase the size of the covariance matrix whose estimation will be worse and entail the risk of constructing

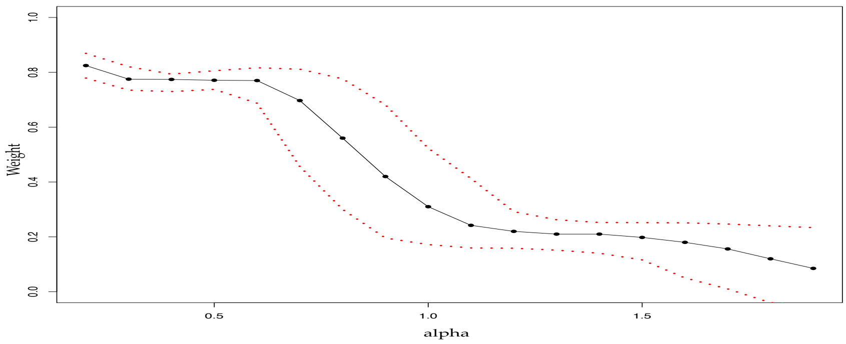
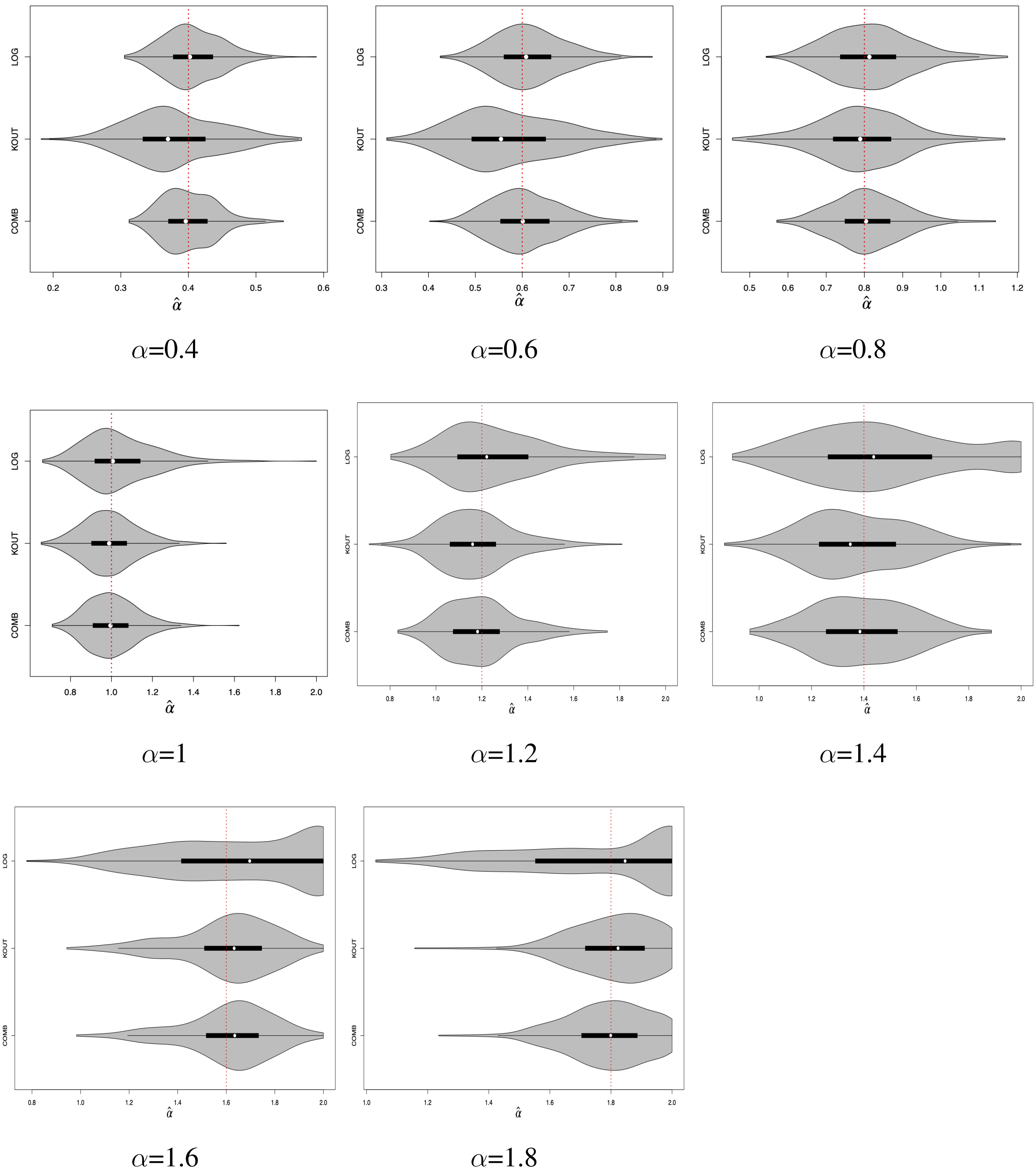


Figure 1: Weight (average) for the LOG-moment estimator $\widehat{\alpha}_{LOG}$ in the combined estimator depending on α. The upper (resp. lower) bound of the interval correspond to the 95% quantile (resp. 5%) for $r = 500$ replications of the combination with $n = 100$ and $B = 1000$.

α	0.2	0.3	0.4	0.5	0.6	0.7
KOUT	$2.4\ 10^{-3}$	$2.5\ 10^{-3}$	$4.6\ 10^{-3}$	$8.3\ 10^{-3}$	$1.3\ 10^{-2}$	$1.5\ 10^{-2}$
LOG	$5.0\ 10^{-4}$	$1.2\ 10^{-3}$	$2.0\ 10^{-3}$	$3.4\ 10^{-3}$	$5.6\ 10^{-3}$	$7.6\ 10^{-3}$
COMB	$4.0\ 10^{-4}$	$8.9\ 10^{-4}$	$1.6\ 10^{-3}$	$2.9\ 10^{-3}$	$5.2\ 10^{-3}$	$6.8\ 10^{-3}$
α	0.8	0.9	1	1.1	1.2	1.3
KOUT	$1.4\ 10^{-2}$	$1.5\ 10^{-2}$	$1.8\ 10^{-2}$	$1.9\ 10^{-2}$	$2.8\ 10^{-2}$	$3.4\ 10^{-2}$
LOG	$1.2\ 10^{-3}$	$2.0\ 10^{-2}$	$3.3\ 10^{-2}$	$4.6\ 10^{-2}$	$5.9\ 10^{-2}$	$7.8\ 10^{-2}$
COMB	$8.3\ 10^{-3}$	$1.1\ 10^{-2}$	$1.5\ 10^{-2}$	$1.8\ 10^{-2}$	$2.6\ 10^{-2}$	$3.1\ 10^{-2}$
α	1.4	1.5	1.6	1.7	1.8	1.9
KOUT	$4.0\ 10^{-2}$	$4.1\ 10^{-2}$	$4.1\ 10^{-2}$	$2.8\ 10^{-2}$	$1.9\ 10^{-2}$	$1.1\ 10^{-2}$
LOG	$8.5\ 10^{-2}$	$8.8\ 10^{-2}$	$9.9\ 10^{-2}$	$8.7\ 10^{-2}$	$7.4\ 10^{-2}$	$7.6\ 10^{-2}$
COMB	$3.3\ 10^{-2}$	$3.4\ 10^{-2}$	$3.5\ 10^{-2}$	$2.5\ 10^{-2}$	$1.9\ 10^{-2}$	$1.1\ 10^{-2}$

Table 4: Mean squared errors for Koutrouvelis regression (KOUT), Log-moment (LOG) and the combined (COMB) estimators of α for $r = 500$, $n = 100$, $B = 1000$ and $\sigma = 1$.

a combined estimator never better than each individual ones. The weight for the log estimator in the combination is represented in Figure 1. We represent in Table 4 the mean squared errors for several values of α. For each value, we remark that the combination between Koutrouvelis and log estimators is always better than each estimator separately. This is confirmed by the plots in Figure 2 comparing the empirical distributions of each estimator.



α=0.4 α=0.6 α=0.8

α=1 α=1.2 α=1.4

α=1.6 α=1.8

Figure 2: Empirical density functions for Log-moment (LOG), Koutrouvelis regression (KOUT) and the combined (COMB) estimators of α for $r = 500$, $n = 100$, $B = 1000$ and $\sigma = 1$.

3 Skewed stable distributions

3.1 Adaptation of estimators for the skewed case

In the case $X \sim S_\alpha(\sigma, \beta, 0)$, we have

$$E[\log|X|] = \left(\frac{1}{\alpha} - 1\right)\gamma + \log\sigma - \frac{\log|\cos\theta|}{\alpha},$$
$$E[(\log|X| - E[\log|X|])^2] = \mathrm{Var}(\log|X|) = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12} - \frac{\theta^2}{\alpha^2},$$

where γ is the Euler constant and $\theta = \arctan\left(\beta\tan\frac{\alpha\pi}{2}\right)$ (see Kuruoglu (2001) prop. 4 and Kateregga et al. (2017) for application).

Let $(X_1, \ldots, X_{2n})$ be a sequence of $2n$ independent and identically distributed stable random variables $S_\alpha(\sigma, \beta, 0)$. We use the centro-symmetrization introduced in Kuruoglu (2001) to the observed data to obtain n independent symmetric stable random variables $S_\alpha(2\sigma, 0, 0)$: $(X_{2k} - X_{2k-1})_{k\in\{1,\ldots,n\}}$. Then, we estimate α by taking then $\widehat{\alpha}_{LOG}^{(n)}(X_2 - X_1, X_4 - X_3, \ldots, X_{2n} - X_{2n-1})$, where $\widehat{\alpha}_{LOG}^{(n)}$ is introduced in Theorem 2.4. In Kuruoglu (2001), the parameter β is also estimated using $\mathrm{Var}[\log|X|]$. He provides a numerical comparison of β estimates.

Another way to estimate α is to use the $(2n-1)$ random variables $(X_k - X_{k-1})_{k\in\{2,\ldots,2n\}}$ by taking $\widehat{\alpha}_{LOG}^{(2n-1)}(X_2 - X_1, X_3 - X_2, \ldots, X_{2n} - X_{2n-1})$. The interest is to preserve the same sample size. However, not enough information is available on the dependence structure of the process $(\log(|X_k - X_{k-1}|))_{k\in N^*}$ to establish the consistency and central limit theorem for this estimate. Given the absence of theoretical result and the numerical comparison provided in Section 3.3, we only consider the first estimate (based on independent increments) in the rest of the study.

3.2 Test of symmetry

We propose a test for checking the skewness of dataset. We want to test H_0 : "$\beta = 0$" against H_1 : "$\beta \neq 0$" using the properties of the estimators studied in the previous section

Let $(X_1, \ldots, X_{2n})$ be a sequence of $2n$ independent and identically distributed stable random variables $S_\alpha(\sigma, \beta, 0)$. Under the null hypothesis H_0, both estimates $\widehat{\alpha}_{LOG}((X_{2k})_k)$ and $\widehat{\alpha}_{LOG}((X_{2k} - X_{2k-1})_k)$ are consistent, and so the difference between these estimators tends to zero. For skewed variables, this convergence does not occur since $\widehat{\alpha}_{LOG}((X_{2k})_k)$ does not converge any more to α. These facts suggest to construct a test based on the difference of these estimators. Denote

$$
\begin{aligned}
L_1 &:= E\left[\log|Z|\right] = \left(\frac{1}{\alpha}-1\right)\gamma + \log\sigma,\\
L_2 &:= E\left[(\log|Z| - E\left[\log|Z|\right])^2\right] = \frac{\pi^2}{6\alpha^2}+\frac{\pi^2}{12},\\
L_3 &:= E\left[(\log|Z| - E\left[\log|Z|\right])^3\right] = 2\zeta(3)\left(\frac{1}{\alpha^3}-1\right),\\
L_4 &:= E\left[(\log|Z| - E\left[\log|Z|\right])^4\right] = \pi^4\left(\frac{3}{20\alpha^4}+\frac{1}{12\alpha^2}+\frac{19}{240}\right),\\
C &:= \mathrm{Cov}\left((\log|X_2| - E[\log|X_2|])^2, (\log|X_2-X_1| - E[\log|X_2-X_1|])^2\right)\\
&\;\;= E\left[(\log|X_2| - E\log|X_2|)^2(\log|X_2-X_1| - E\log|X_2-X_1|)^2\right] - L_2^2,
\end{aligned}
$$

where Z is an $S_\alpha(\sigma,0,0)$ random variable, ζ is the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ and $\zeta(3)=1.2020569\ldots$.

Proposition 3.1. *Denote $\widehat{\alpha}_{LOG}(\{X_{2k}\})$ and $\widehat{\alpha}_{LOG}(\{X_{2k}-X_{2k-1}\})$ the log-moment estimate calculated respectively on the samples $(X_{2k})_{k=1,\ldots,n}$ and $(X_{2k}-X_{2k-1})_{k=1,\ldots,n}$. For $w\in(0,1)$, define the critical region*

$$
R_w = \left\{ (x_1,...,x_{2n})\in\mathbb{R}^{2n} \;\middle|\; n\frac{\left[\widehat{\alpha}_{LOG}(\{x_{2k}\}) - \widehat{\alpha}_{LOG}(\{x_{2k}-x_{2k-1}\})\right]^2}{18\pi^{-4}\widehat{\alpha}^6_{LOG}(\{x_{2k}-x_{2k-1}\})(\widehat{L_4}-\widehat{L_2}^2-\widehat{C})} > t_w \right\},
$$

where t_w is the $1-w$ quantile of the Chi-squared distribution with 1 degree of freedom, and where $\widehat{L_4}$, $\widehat{L_2}$ and $\widehat{C}$ are respectively the empirical moments of L_4, L_2 and C. We decide to reject the null hypothesis if $(X_1,\ldots,X_{2n})\in R_w$. The test has an asymptotic significance level equal to w and is asymptotically consistent under H_1.

Proof. We denote $Y_k=\log|X_{2k}|$ and $Z_k=\log|X_{2k}-X_{2k-1}|$ for $k=1,\ldots,n$. Under the null hypothesis, we have

$$
\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n}\begin{pmatrix}(Y_k-\overline{Y_n})^2\\(Z_k-\overline{Z_n})^2\end{pmatrix} - \begin{pmatrix}L_2\\L_2\end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}L_4-L_2^2 & C\\ C & L_4-L_2^2\end{pmatrix}\right).
$$

Then by multidimensional delta method, we get

$$
\sqrt{n}\left(\begin{pmatrix}\widehat{\alpha}_{LOG}((X_{2k})_k)\\ \widehat{\alpha}_{LOG}((X_{2k}-X_{2k-1})_k)\end{pmatrix} - \begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \frac{9\alpha^6}{\pi^4}\begin{pmatrix}L_4-L_2^2 & C\\ C & L_4-L_2^2\end{pmatrix}\right)
$$

and

$$
\sqrt{n}\frac{\widehat{\alpha}_{LOG}((X_{2k})_k) - \widehat{\alpha}_{LOG}((X_{2k}-X_{2k-1})_k)}{\sqrt{\frac{18\alpha^6}{\pi^4}(L_4-L_2^2-C)}} \xrightarrow{d} \mathcal{N}(0,1).
$$

Finally, applying the Slutsky theorem and the consistency of $\widehat{L_4}$, $\widehat{L_2}$ and $\widehat{C}$, we obtain that the asymptotic significance level is equal to w.

Under H_1, we have

$$\widehat{\alpha}_{LOG}((X_{2k})_k) - \widehat{\alpha}_{LOG}((X_{2k} - X_{2k-1})_k) \xrightarrow{a.s.} \frac{\pi\alpha}{\sqrt{\pi^2 - 6\theta^2}} - \alpha \neq 0$$

with $\theta = \arctan\left(\beta \tan \frac{\alpha\pi}{2}\right)$. Then, under the alternative $\beta \neq 0$ we have a consistent test, $P(R_w) \xrightarrow[n\to\infty]{} 1$. □

3.3 Numerical performances

Estimation We compare the performance of both log-moment estimates obtained after symmetrization. The mean squared errors of the first estimate calculated on $(X_{2k} - X_{2k-1})_{k\in\{1,\ldots,n\}}$ do not depend on β since $(X_{2m} - X_{2m-1})$ are symmetric and independent. We compare this estimate with the second estimate calculated on $(X_k - X_{k-1})_{k\in\{2,\ldots,2n\}}$. The joint distribution of $(X_k - X_{k-1})_{k\in\{2,\ldots,2n\}}$ is unknown, and could depend on β. However, we observe numerically that its mean squared error does not depend on β. Table 5 provides a comparison of both two estimates in term of mean squared error. For $\alpha < 1$, both estimates have similar performance despite a 2-fold on sample sizes, this shows that the dependence degrades the precision. This impact of dependence is less pronounced for $\alpha \geq 1$. We improve the precision by taking the estimate on dependent sample. However, Koutrouvelis method always outperforms it when $\alpha > 1$ (see Tables 2 and 4).

The comparison can be done on these tables for $\beta = 0$ since Koutrouvelis method does not vary with skewed distributions. Indeed, the modulus of the characteristic function depends only on α and σ.

In light of these numerical results, we decide to combine the log-moment estimate after symmetrization $\widehat{\alpha}_{LOG}(\{X_{2k} - X_{2k-1}\}_{k=1,\ldots,n})$ (named after LOG sym.) with the Koutrouvelis estimator in the same way that with symmetric variables (see Section 2.2) to obtain a new estimator whose numerical performances are reported on Table 6.

For skewed data (see Table 6), the combined estimators still have good performance but we loose in term of mean squared errors comparing to the symmetric case.

Testing procedure We use Monte Carlo experiments to evaluate the empirical significance level, that is the probability to reject the null hypothesis H_0 under H_0. Table 7 gives the performances of our testing procedure under the null hypothesis. For small values of α, the empirical significance level converges slowly to w: this is due to the form of the density that is concentrated around zero, and the poor quality of the estimation of the coefficient $L_4 - L_2^2 - C$. The estimation converges rather slowly to this coefficient (which increases when α decreases).

To evaluate the performance of the test, we also examine the distribution of the p-values. Under the null hypothesis, the p-value converges in distribution to the uniform distribution on

$2n$		$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
100	n obs. i.i.d	$1.01\ 10^{-3}$	$4.44\ 10^{-3}$	$1.27\ 10^{-2}$	$3.30\ 10^{-2}$	$10.7\ 10^{-2}$
	$2n-1$ obs. dep.	$0.970\ 10^{-3}$	$4.14\ 10^{-3}$	$1.08\ 10^{-2}$	$2.44\ 10^{-2}$	$6.08\ 10^{-2}$
500	n obs. i.i.d	$1.80\ 10^{-4}$	$7.90\ 10^{-4}$	$2.10\ 10^{-3}$	$4.68\ 10^{-3}$	$1.01\ 10^{-2}$
	$2n-1$ obs. dep.	$1.75\ 10^{-4}$	$7.58\ 10^{-4}$	$1.86\ 10^{-3}$	$3.81\ 10^{-3}$	$7.33\ 10^{-3}$
1000	n obs. i.i.d	$9.12\ 10^{-5}$	$3.86\ 10^{-4}$	$9.90\ 10^{-4}$	$2.23\ 10^{-3}$	$4.79\ 10^{-3}$
	$2n-1$ obs. dep.	$8.94\ 10^{-5}$	$3.69\ 10^{-4}$	$8.88\ 10^{-4}$	$1.84\ 10^{-3}$	$3.48\ 10^{-3}$

$2n$		$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
100	n obs. i.i.d	$12.6\ 10^{-2}$	$14.4\ 10^{-2}$	$12.0\ 10^{-2}$	$9.76\ 10^{-2}$
	$2n-1$ obs. dep.	$8.78\ 10^{-2}$	$11.2\ 10^{-2}$	$9.59\ 10^{-2}$	$7.47\ 10^{-2}$
500	n obs. i.i.d	$2.14\ 10^{-2}$	$4.32\ 10^{-2}$	$5.46\ 10^{-2}$	$4.76\ 10^{-2}$
	$2n-1$ obs. dep.	$1.35\ 10^{-2}$	$2.51\ 10^{-2}$	$3.61\ 10^{-2}$	$3.31\ 10^{-2}$
1000	n obs. i.i.d	$9.98\ 10^{-3}$	$2.02\ 10^{-2}$	$3.35\ 10^{-2}$	$3.20\ 10^{-2}$
	$2n-1$ obs. dep.	$6.51\ 10^{-3}$	$1.17\ 10^{-2}$	$2.02\ 10^{-2}$	$2.18\ 10^{-2}$

Table 5: Mean squared errors for α using log-moments for $2n$ random variables i.i.d. $S_\alpha(1, \beta, 0)$.

α	0.2	0.4	0.6	0.8	1
COMB	$2.39\ 10^{-3}$	$2.21\ 10^{-3}$	$5.88\ 10^{-3}$	$9.08\ 10^{-3}$	$1.67\ 10^{-2}$
KOUT	$6.14\ 10^{-3}$	$4.92\ 10^{-3}$	$1.10\ 10^{-2}$	$1.36\ 10^{-2}$	$1.72\ 10^{-2}$
LOG sym.	$1.90\ 10^{-3}$	$4.36\ 10^{-3}$	$1.34\ 10^{-2}$	$2.67\ 10^{-2}$	$7.29\ 10^{-2}$

α	1.2	1.4	1.6	1.8
COMB	$2.70\ 10^{-2}$	$3.53\ 10^{-2}$	$3.54\ 10^{-2}$	$2.20\ 10^{-2}$
KOUT	$2.71\ 10^{-2}$	$4.15\ 10^{-2}$	$4.16\ 10^{-2}$	$2.41\ 10^{-2}$
LOG sym.	$1.24\ 10^{-1}$	$1.38\ 10^{-1}$	$1.29\ 10^{-1}$	$1.10\ 10^{-1}$

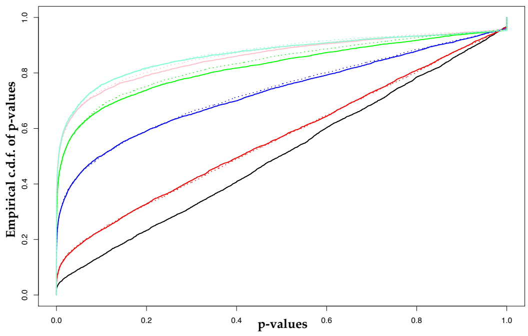
Table 6: Mean squared errors for the combined (COMB), Koutrouvelis regression (KOUT) and Log-moment (LOG) estimators of α for $r = 500$, $n = 100$, $B = 1000$, $\beta = 0.6$ and $\sigma = 1$.

$[0, 1]$. Under alternative, the p-value converges in probability to zero, and the converge rate indicates the power of the test. In Figure 3, we can thus see that the power increases when β goes away from 0 or when the sample sizes increases. This convergence under the alternative hypothesis depends on the values of α and β. The case $\beta = 0$ confirms that the empirical significance level converges quickly to the nominal level ω.

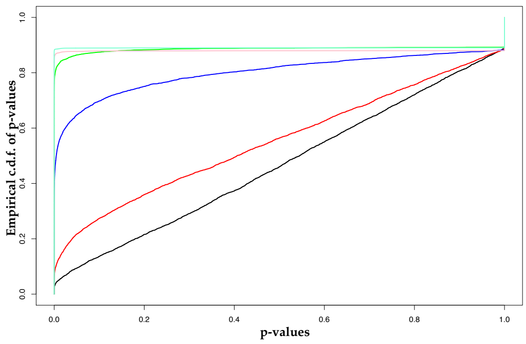
We observe that the p-value takes the value 1 with non-null probability for small size of samples. This jump is due to the truncation in the log-moment estimator defined in Theorem 2.4. Indeed, the truncation imposed on α to be equal to 2, thus the distribution is symmetric and we always accept the null hypothesis H_0. This phenomenon asymptotically disappears since the estimate is consistent.

	$\alpha = 0.6$	0.8	1	1.2	1.4	1.6	1.8
$2n = 200$	0.084	0.087	0.088	0.097	0.088	0.06	0.042
$2n = 1000$	0.097	0.089	0.074	0.065	0.067	0.061	0.035
$2n = 10^4$	0.10	0.071	0.051	0.050	0.048	0.051	0.053
$2n = 5\ 10^4$	0.081	0.065	0.050	0.050	0.049	0.050	0.051

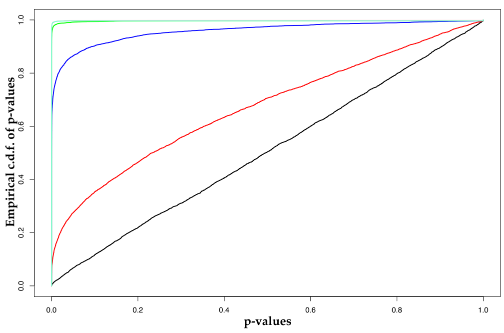
Table 7: Probabilities to reject the null hypothesis under H_0 for several sizes of samples and different values of α. The significance level is $w = 5\%$.



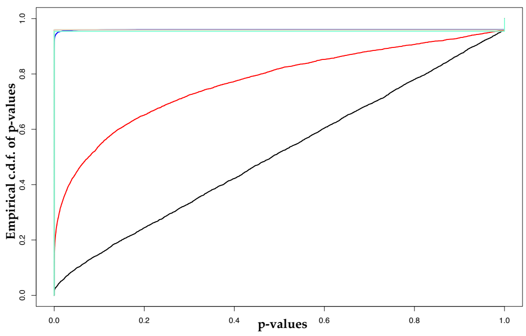
(a) $2n = 200$, $\alpha = 1.2$



(b) $2n = 200$, $\alpha = 0.8$



(c) $2n = 1000$, $\alpha = 1.2$



(d) $2n = 1000$, $\alpha = 0.8$

Figure 3: Representation of the empirical cumulative distribution function of p-values for different values of β, $\beta = 0$ (black), $\beta = 0.2$ (red), $\beta = 0.4$ (dark blue), $\beta = 0.6$ (green), $\beta = 0.8$ (pink) and $\beta = 1$ (light blue). In (a), we add in dotted lines the value for $\beta \in \{-1, -0.8, -0.6, -0.4, -0.2\}$ which correspond exactly to the positive ones.

4 Case of non-identically distributed stable variables

In applications, it may be the case that one needs to analyze non-stationary phenomena. For instance, it seems plausible that financial logs which display jumps will see the intensity of these jumps depend on external events, such as crises (see next section for an illustration on the S&P 500). Sometimes, the variation of α will be slow, and it is of interest to investigate under which conditions our estimator still behaves correctly in situations where the data at hand deviate slightly from the assumption that the random variables have the same distribution. In the sequel, we examine two cases: deterministic and random small perturbations of α, leading to random variables which are not identically distributed. We do not dispense here with the independence assumption, although this would be a desirable extension. This generalization will be useful to address the estimation of the stability function for the self-stabilizing processes (see Falconer and Lévy Véhel (2018a,b)).

4.1 Deterministic perturbations

Let $(X_i)_i$ be a sequence of independent variables and X random variable independent of $(X_i)_i$ such that $X_i \sim S_{\alpha_i}(\sigma_i, 0, 0)$ and $X \sim S_\alpha(\sigma, 0, 0)$. We denote $Y_i = \log|X_i|$, $Y = \log|X|$. Assume there are constants $(c_\alpha, c_\sigma) \in (0,1)^2$ such that for each integer i,

$$\alpha_i = \alpha + \varepsilon_i \in (0, 2] \text{ and } \sigma_i = \sigma + \eta_i,$$

with ε_i and η_i deterministic satisfying

$$\frac{|\varepsilon_i|}{\alpha} \leq c_\alpha < 1 \text{ and } \frac{|\eta_i|}{\sigma} \leq c_\sigma < 1.$$

Proposition 4.1. *Under the conditions* $\displaystyle\frac{1}{n}\sum_{i=1}^{n}|\varepsilon_i| \xrightarrow[n\to\infty]{} 0$ *and* $\displaystyle\frac{1}{n}\sum_{i=1}^{n}|\eta_i| \xrightarrow[n\to\infty]{} 0$, *one has*

$$\widehat{\alpha}_{LOG}^{(n)} \xrightarrow[n\to\infty]{a.s.} \alpha,$$

where $\widehat{\alpha}_{LOG}^{(n)}$ *is defined in Theorem 2.4.*

Proof. See Appendix. $\square$

Proposition 4.2. *Let* $\Sigma_{\alpha,\sigma}$ *be the covariance matrix between* Y *and* Y^2*:*

$$\Sigma_{\alpha,\sigma} = \begin{pmatrix} \mathrm{Var}(Y) & \mathrm{Cov}(Y,Y^2) \\ \mathrm{Cov}(Y,Y^2) & \mathrm{Var}(Y^2) \end{pmatrix} \tag{4.1}$$

and set

$$H_{\alpha,\sigma} = \begin{pmatrix} 6\left(\left(\frac{1}{\alpha}-1\right)\gamma+\log\sigma\right)\alpha^3/\pi^2 \\ -3\alpha^3/\pi^2 \end{pmatrix}. \tag{4.2}$$

With the conditions $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}|\varepsilon_i| \xrightarrow[n\to\infty]{} 0$ *and* $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}|\eta_i| \xrightarrow[n\to\infty]{} 0$*, the following central limit theorem holds for* $\widehat{\alpha}_{LOG}^{(n)}$*:*

$$\sqrt{n}\left(\widehat{\alpha}_{LOG}^{(n)} - \alpha\right) \xrightarrow{d} \mathcal{N}(0,\ H_{\alpha,\sigma}^{\top}\Sigma_{\alpha,\sigma}H_{\alpha,\sigma}). \tag{4.3}$$

Proof. See Appendix. □

4.2 Random perturbations

Let X be a random variable with a stable distribution $S_\alpha(\sigma,0,0)$. For each integer i, denote $\alpha_i = \alpha + \varepsilon_i$ where ε_i is a random variable. Suppose there is a constant $c_\alpha \in (0,1)$ such that

$$P\left(\frac{\varepsilon_i}{\alpha} \in \left[-c_\alpha, \min\left(c_\alpha, \frac{2}{\alpha} - 1\right)\right]\right) = 1.$$

Let $(X_i)_i$ be a sequence of independent variables and independent of X such that $X_i \sim S_{\alpha_i}(\sigma,0,0)$ (given α_i). We denote $Y = \log|X|$ and $Y_i = \log|X_i|$ for $i \in \mathbb{N}$.

Proposition 4.3. *Under the conditions* $\displaystyle\frac{1}{n}\sum_{i=1}^{n} E[|\varepsilon_i|] \xrightarrow[n\to\infty]{} 0$*, we have*

$$\widehat{\alpha}_{LOG}^{(n)} \xrightarrow[n\to\infty]{a.s.} \alpha.$$

If, in addition, $\displaystyle\frac{1}{\sqrt{n}}\sum_{i=1}^{n} E[|\varepsilon_i|] \xrightarrow[n\to\infty]{} 0$*, then the following central limit theorem holds:*

$$\sqrt{n}\left(\widehat{\alpha}_{LOG}^{(n)} - \alpha\right) \xrightarrow{d} \mathcal{N}(0, H_{\alpha,\sigma}\Sigma_{\alpha,\sigma}\ H_{\alpha,\sigma}^{\top})$$

where $\Sigma_{\alpha,\sigma}$ *and* $H_{\alpha,\sigma}$ *are defined in* (4.1) *and* (4.2).

Proof. See Appendix. □

5 Some applications for the combined estimator

In this section, we apply the combined estimate to processes with varying stable index. The estimate is calculated on small window with respect to the number of observations to catch the variations of α. This empirical study shows that the performance of the combined estimator for small samples makes the local estimation of α possible.

5.1 Numerical results on synthetic data: multistable Lévy motion

We now use our log-moment and combined estimators in the case of the multistable Lévy motion defined in Falconer and Lévy Véhel (2009) (see also Le Guével and Lévy Véhel (2012) for further properties of this process). The basic idea is to allow the stability index evolve with time, so that the jump intensity, which is governed by α, varies along a trajectory. Such a feature is commonly encountered in times series observed in fields such as finance or biomedicine (see for instance Corlay et al. (2014); Frezza (2018); Fischer et al. (2003); Bianchi et al. (2013)). Let us briefly recall the definition of such processes.

Let $\alpha : [0,1] \to (0,2)$ be continuously differentiable. We denote $r^{<s>} = \text{sign}(r)|r|^s$ for $r \in \mathbb{R}$ and $s \in \mathbb{R}$. Symmetric multistable Lévy motion is defined by

$$M_\alpha(t) = C_{\alpha(t)} \sum_{(X,Y)\in\Pi} 1_{(0,t]}(X) Y^{<-1/\alpha(t)>} \tag{5.1}$$

where $C_\theta = \left(\int_0^\infty u^{-\theta} \sin(u) du \right)^{-1/\theta}$ and Π is a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}$ with plane Lebesgue measure $\mathcal{L}^2$ as mean measure. This process is simulated by using the field

$$X(t,u) = C_{\alpha(u)} \sum_{(X,Y)\in\Pi} 1_{(0,t]}(X) Y^{<-1/\alpha(u)>}.$$

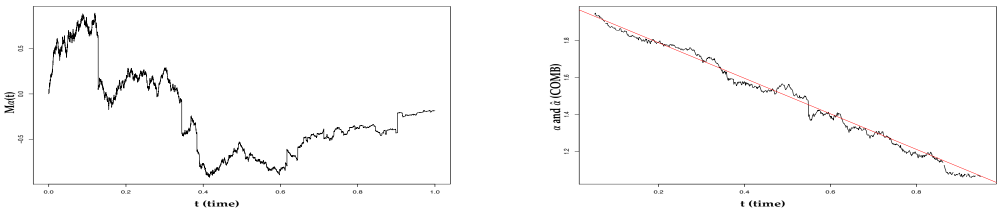
For each $u \in (0,1)$, $X(.,u)$ is an $\alpha(u)$-stable process with independent increments which can be implemented using the RSTAB program available in Stoev and Taqqu (2004) or in Samorodnitsky and Taqqu (1994). The interval $[0,1]$ is discretized in N equal parts and $X(.,u)$ is implemented by the cumulative sum of N independent stable random variables with $\alpha(u)$ as characteristic exponent.

In Figure 4, we display sample paths of multistable processes for several α functions. Then, we estimate these functions at all point t_0 thanks to the combined estimator with a window of n observations around t_0. Therefore, from a realization $\left(M_\alpha(\frac{k}{N})\right)_{k=0,..,N}$ of a multistable process, the function α can only be estimated on the interval $[\frac{n}{N}; 1 - \frac{n}{N}]$.

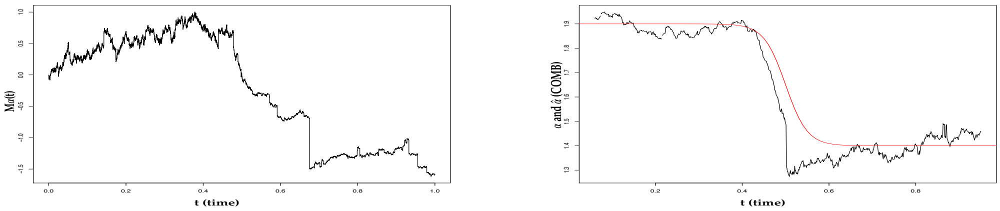
In Figure 5, we iterate 100 times the simulation and the estimation for a multistable process with $\alpha(t) = 1.5 - 0.48 \sin(2\pi(t + 1/4))$. For each point where the function α is estimated, we obtain the empirical distribution of the combined estimator. This procedure is repeated for several sizes of window (100, 200, 1000 and 2000). We observe that the standard error which corresponds to the standard deviation for the combined estimator is decreasing when the size of the window n increases whereas the bias is increasing for n large. Finally, the mean squared error is decreasing when n increases until $n = 1000$ and then increases for larger value. Figure 6 represents the bias, standard error and mean squared error as function of time t for various values of the window size.

The mean squared error as a function of α is reported in Figure 7. The mean squared error does not vary much according to the value of α in $[1,2]$ when n is fixed.

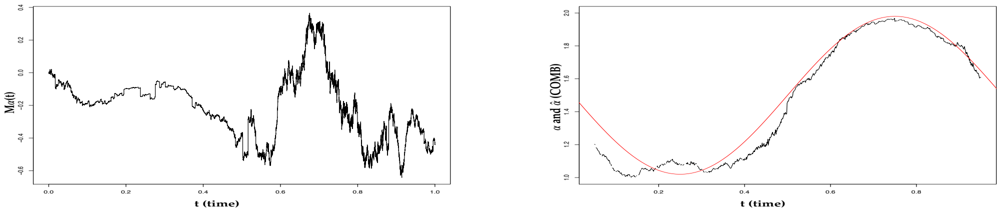
As these figures show, reasonable estimates are obtained on these experiments, due to the fact that the variations of α are "slow" compared to the sampling frequency: this feature ensures that centering a window around any given t_0 and treating all points inside this window as having



$\alpha(t) = 1.98 - 0.96t$

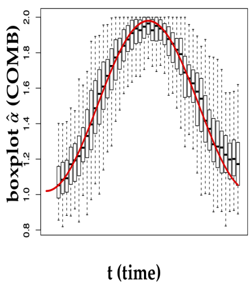


$\alpha(t) = 1.98 - \frac{0.96}{1+exp(20-40t)}$

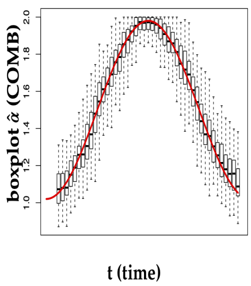


$\alpha(t) = 1.5 - 0.48\sin(2\pi t)$

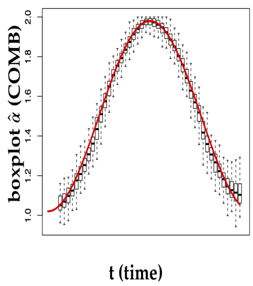
Figure 4: Trajectories of multistable processes on (0,1) with $N = 20000$ points in the first column. The functions $\alpha(t)$ (red) and $\widehat{\alpha}_{COMB}(t)$ (black) are represented in the second column with $n = 2000$.



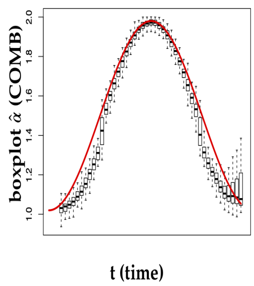
(a) $n = 100$



(b) $n = 200$



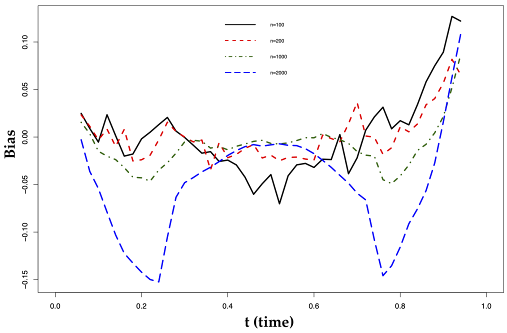
(c) $n = 1000$



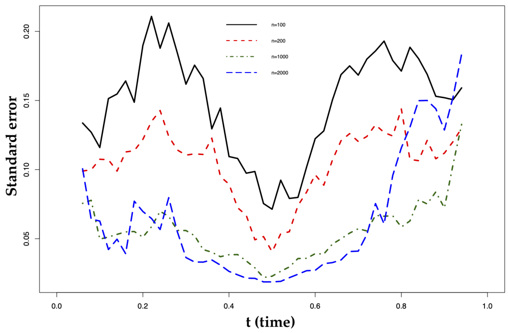
(d) $n = 2000$

Figure 5: Box-plots of the estimator $\widehat{\alpha}^{(n)}_{COMB}$ for 100 replications of a multistable process with characteristic exponent $\alpha(t) = 1.5 - 0.48\sin(2\pi(t + 1/4))$, represented in red. The box-plots represent the behavior of the estimator for several sizes n of window.

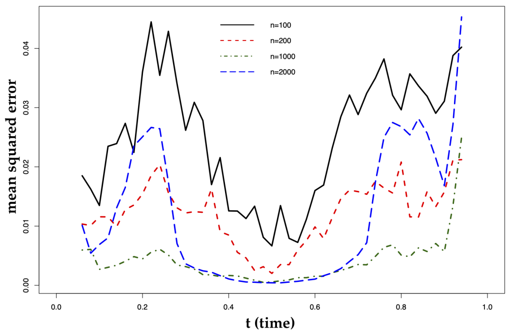
the same α value is an acceptable approximation as far as estimation is concerned.



(a)



(b)



(c)

Figure 6: Representation of the bias (a), standard error (b) and mean squared error (c) as function of t for $n = 100$ (black solid line), $n = 200$ (red dashed), $n = 1000$ (green dotted) and $n = 2000$ (blue dotted and dashed mix.). The statistics are evaluated on the same trajectories in Figure 5.

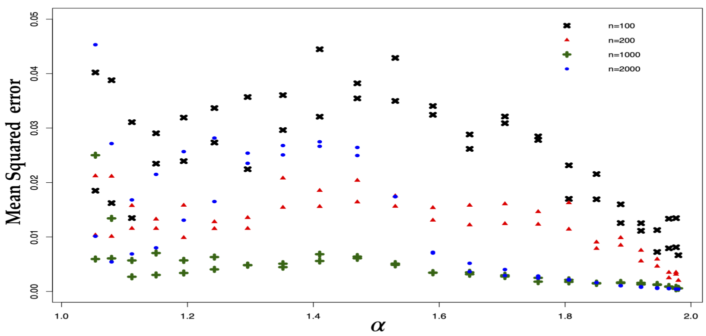


Figure 7: Mean squared error according to the value of α of the combined estimation for a multistable process with $\alpha(t) = 1.5 - 0.48\sin(2\pi(t + 1/4))$. The statistics are evaluated on the same trajectories as Figure 5.

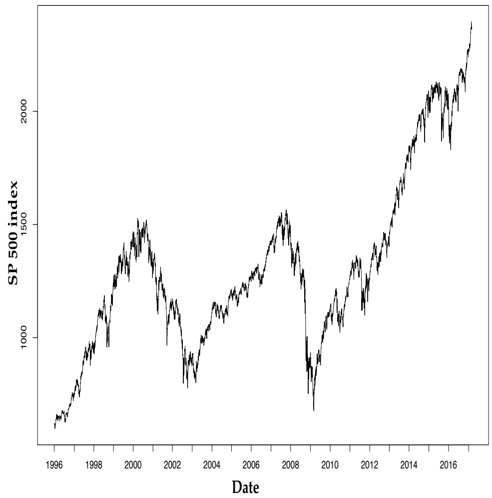
5.2 Application on financial logs

This last section deals with real data. We want to apply the combined estimator to estimate the characteristic exponent of the financial index Standard $\&$ Poor's 500 (abbreviated as the *S&P 500*, see Figure 8). This is a stock market index based on the 500 companies having largest capitalization in the United States. The stock market returns of the financial index *S&P 500*, which correspond to the renormalized growth rate $((Y_{t+1} - Y_t)/Y_t)_t$, are supposed to be independent stable random variables. We analyze the behavior of the index around the Wall Street Crash of 1929, and during the period 1996-2017. For both periods, we test the symmetry and we estimate the exponent α in sliding window.

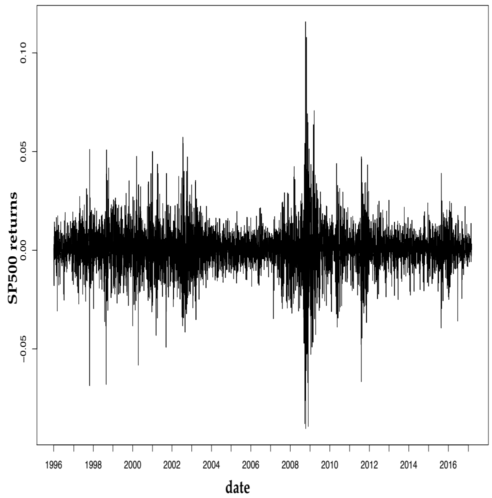
Period 1996-2017 In Figure 9 (a), we first test the symmetry of the data in sliding window of size 1000 (using the test defined in Proposition 3.1). We represent the empirical distribution function for the p-values calculated on the sliding windows since 1996. The cumulative distribution function is very close to the uniform one. This correspond to the p-values distribution under the null hypothesis. Then, the symmetric hypothesis is not rejected for S&P 500 returns since 1996. As a consequence, the parameter is estimated by applying the estimator defined in Theorem 2.4 in sliding window of several sizes during the period 1996-2017 (see Figure 10). We observe the effect of the window size on the estimation of the function α. The overall pattern of the three estimates is the same. However, the regularity of the estimated function varies, it is a classical problem of bias-variance trade-off. When the sample size is fixed, we do not have theoretical results on bias and variance for non identically distributed models. These results would allow to propose a choice of window that satisfies the bias-variance trade-off. In practice, the estimation of the bias and the variance is a difficult problem that requires generally prior information on the α function. If the variations of α are quite low over small intervals, the bias and the variance could be evaluated for example using a bootstrap method.

Wall Street Crash of 1929 In Figure 9(b), the distribution of the p-value allows to reject the hypothesis of symmetry between 1929 and 1936. In Figure 11, the estimation of the characteristic exponent is done between 1929 and 1936 using the skewed combined estimator defined in Section 3. A sudden drop is observed at the end of 1929. This change corresponds to the Wall Street financial crash of 1929. The estimation for symmetric data (in red) is added in the figure to see the difference between the two estimations, particularly during the crisis.

The empirical comparison of these two periods highlights the effect of a financial crisis on the two parameters α and β. Indeed, the crisis has an impact on the jump activity but it also create an asymmetric situation.

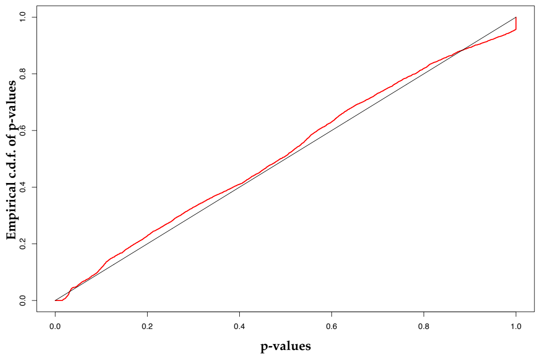


(a)

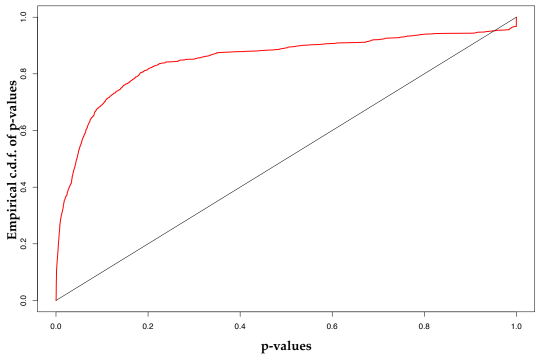


(b)

Figure 8: Evolution of the financial index S&P 500 as function of t (a) and its return (b), between 1996 and 2017.

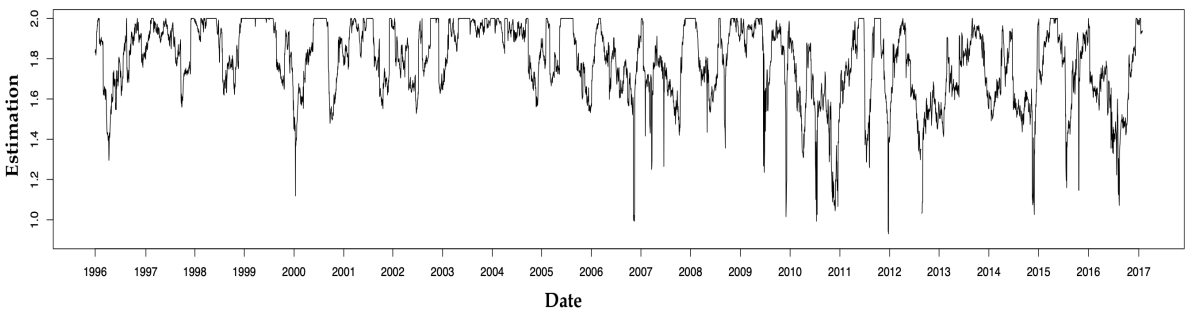


(a) 1996-2017

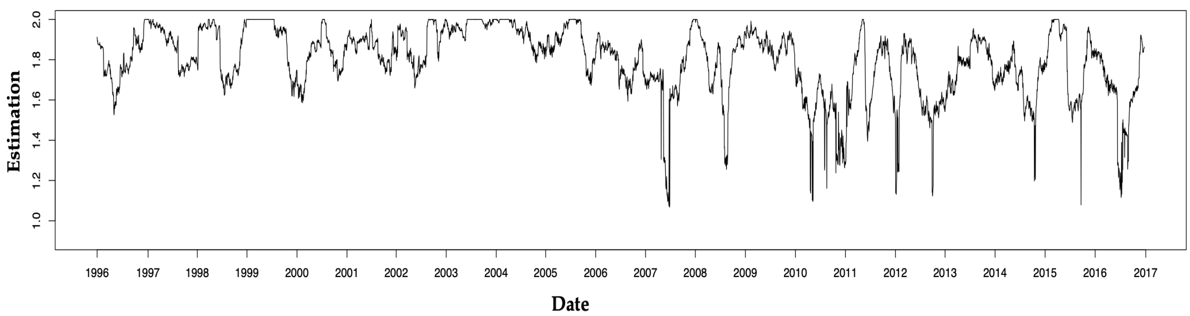


(b) 1928-1936

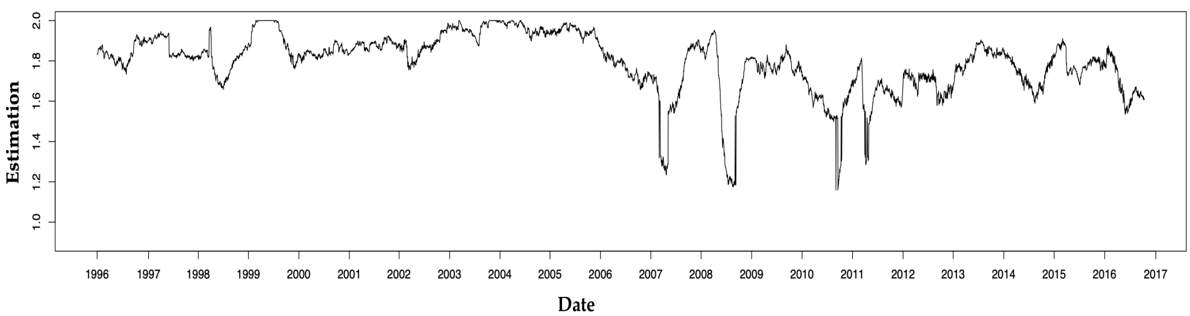
Figure 9: Representation of the empirical cumulative distribution function of the p-value for S&P 500 returns between 1996 and 2017 (Figure (a)) and between 1928 and 1936 (Figure (b)) with sliding window of size 1000.



(a) n=50



(b) n=100



(c) n=200

Figure 10: Values of the combined estimator $\widehat{\alpha}_{COMB}^{(n)}$ for the S&P 500 characteristic exponent α in sliding window of several sizes n for working days between 1996 and 2017.

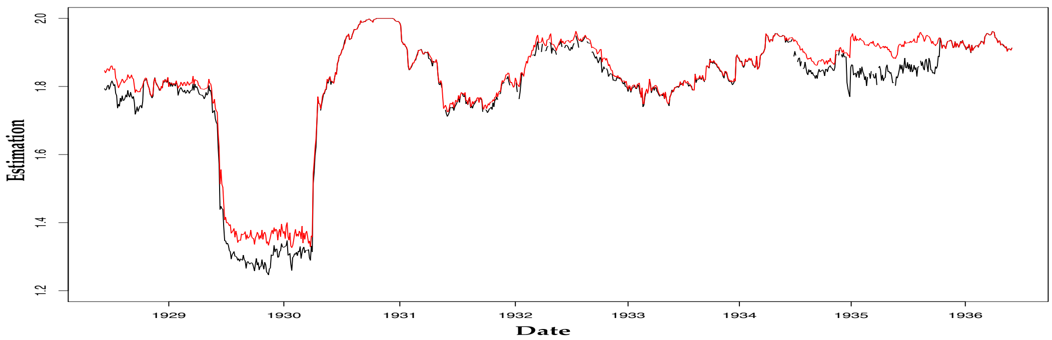


Figure 11: Values of the skewed combined estimator $\widehat{\alpha}^{(n)}_{COMB}$ (in black) and the symmetric combined estimator (in red) for the S&P 500 characteristic exponent α around the Wall Street financial crash of 1929 (sliding window of size $n = 200$ observations).

6 Appendix

Proof of Proposition 2.1

Let Z be a stable random variable $S_\alpha(1,0,0)$. It satisfies the following properties:

- Z has bounded density,
- for $\alpha \in (0,2)$, the asymptotic behavior of the tail probabilities is

$$\lim_{\lambda\to+\infty} \lambda^\alpha P(Z>\lambda) = \lim_{\lambda\to+\infty} \lambda^\alpha P(Z<-\lambda) = \frac{1}{2}C_\alpha, \tag{6.1}$$

  where $C_\alpha = \left(\int_0^\infty u^{-\alpha}\sin(u)du\right)^{-1/\alpha}$ (see Samorodnitsky and Taqqu (1994), Property 1.2.15).

Thus we deduce for all $p>0$,

$$E[|\log|Z||^p] = \int_0^\infty \mathbb{P}(|\log|Z||^p > x)dx < \infty.$$

For the gaussian case $\alpha = 2$, this result is immediate.

Proof of Proposition 2.2

We use the formula of Property 1.2.17 and 1.2.15 in Samorodnitsky and Taqqu (1994) to compute $E[|Z|^t]$ for $0<t<\alpha$:

$$E[|Z|^t] = \frac{2^{t-1}\Gamma(1-t/\alpha)}{t\int_0^{+\infty}\frac{\sin^2(u)}{u^{t+1}}du} = \frac{2^{t-1}\Gamma(1-t/\alpha)}{\int_0^{+\infty}\frac{\sin(2u)}{u^{t}}du} = \frac{(1-t)\Gamma(1-t/\alpha)}{\Gamma(2-t)\cos(\pi t/2)}.$$

Furthermore, if $0<t<1$, we have:

$$E[|Z|^t] = \frac{\Gamma(1-t/\alpha)}{\Gamma(1-t)\cos(\pi t/2)}.$$

Proof of Theorem 2.3

By the strong law of large numbers for the random variables $(\log|X_i|)_i$ with finite expectation (Proposition 2.1), and the continuous mapping theorem for $g(x) = \frac{\gamma}{\gamma+x}$, we get $\widehat{\alpha}_n \xrightarrow[n\to\infty]{a.s.} \alpha$. Then we apply the central limit theorem

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \log|X_i| - (\alpha^{-1}-1)\gamma\right) \xrightarrow{d} \mathcal{N}(0, f(\alpha)),$$

where $f(x) = \frac{\pi^2}{6x^2} + \frac{\pi^2}{12}$. Using the delta method with the function g, we obtain

$$\sqrt{n}(\widehat{\alpha}_n - \alpha) \xrightarrow{d} \mathcal{N}(0, f(\alpha)\alpha^4\gamma^{-2}).$$

Finally, Slutsky's theorem leads to the following result:

$$\frac{\sqrt{n}(\widehat{\alpha}_n - \alpha)\gamma}{\widehat{\alpha}_n^2\sqrt{f(\widehat{\alpha}_n)}} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof of Theorem 2.4

The proof is based on the same arguments as Theorem 2.3.

The random variables $(\log|X_i|)_i$ have finite second moment, so we can apply the continuous mapping theorem with $x \mapsto \frac{\pi}{\sqrt{6x-\pi^2/2}}$ to the empirical variance of $(\log|X_i|)_i$. We get the strong consistency of $\hat{\alpha}_{LOG}^{(n)}$. For $\hat{\sigma}_{LOG}^{(n)}$, it is similar with the function $(x,y) \mapsto e^{x-(y^{-1}-1)\gamma}$ applied to $(\frac{1}{n}\sum_{i=1}^n \log|X_i|\ ;\ \hat{\alpha}_{LOG}^{(n)})$.

Then, we need the third and the fourth log-moment of a stable law $Z \sim S_\alpha(1,0,0)$ for the covariance matrix. We have

$$E\left[(\log|Z| - E\left[\log|Z|\right])^3\right] = 2\zeta(3)\left(\frac{1}{\alpha^3} - 1\right)$$

and we get

$$E[(\log|Z|)^3] = \frac{4\gamma^3 + 2\gamma\pi^2 + 8\zeta(3)}{4\alpha^3} + \frac{-12\gamma^3 - 2\gamma\pi^2}{4\alpha^2} + \frac{12\gamma^3 + \gamma\pi^2}{4\alpha} + \frac{-4\gamma^3 - \gamma\pi^2 - 8\zeta(3)}{4}$$

and

$$E\left[(\log|Z| - E\left[\log|Z|\right])^4\right] = \pi^4\left(\frac{3}{20\alpha^4} + \frac{1}{12\alpha^2} + \frac{19}{240}\right) \tag{6.2}$$

$$\begin{aligned}E[(\log|Z|)^4] = &\frac{240\gamma^4 + 240\gamma^2\pi^2 + 36\pi^4 + 1920\zeta(3)\gamma}{240\alpha^4} + \frac{-960\gamma^4 - 480\gamma^2\pi^2 - 1920\zeta(3)\gamma}{240\alpha^3}\\ &+ \frac{1440\gamma^4 + 360\gamma^2\pi^2 + 20\pi^4}{240\alpha^2} + \frac{-960\gamma^4 - 240\gamma^2\pi^2 - 1920\zeta(3)\gamma}{240\alpha}\\ &+ \frac{240\gamma^4 + 120\gamma^2\pi^2 + 19\pi^4 + 1920\zeta(3)\gamma}{240}\end{aligned}$$

where ζ is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ and $\zeta(3) = 1.2020569\ldots$.

According to the multivariate central limit theorem we have

$$\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n \log|X_i| \\ \frac{1}{n}\sum_{i=1}^n (\log|X_i|)^2 \end{pmatrix} - \begin{pmatrix} E(\log|X_1|) \\ E[(\log|X_1|)^2] \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\alpha,\sigma}),$$

where $\Sigma_{\alpha,\sigma}$ is defined in (2.9). Then, we apply twice the delta method. First with the function $(x,y) \mapsto \left(x\ ;\ \frac{\pi}{\sqrt{6(y-x^2)-\pi^2/2}}\right)$, we get

$$\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n \log|X_i| \\ \hat{\alpha}_{LOG}^{(n)} \end{pmatrix} - \begin{pmatrix} E(\log|X_1|) \\ \alpha \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}(0, G_{\alpha,\sigma}\Sigma_{\alpha,\sigma}G_{\alpha,\sigma}^\top).$$

where $G_{\alpha,\sigma}$ is defined in (2.8). Then with the function $(x,y) \mapsto \left(e^{x-(y^{-1}-1)\gamma}\ ;\ y\right)$, we obtain

$$\sqrt{n}\left(\begin{pmatrix} \hat{\sigma}^{(n)}_{LOG} \\ \hat{\alpha}^{(n)}_{LOG} \end{pmatrix} - \begin{pmatrix} \sigma \\ \alpha \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}(0, F_{\alpha,\sigma} G_{\alpha,\sigma} \Sigma_{\alpha,\sigma} G^\top_{\alpha,\sigma} F^\top_{\alpha,\sigma}),$$

where $F_{\alpha,\sigma}$ is defined in (2.8). This concludes the proof.

Proof of Proposition 4.1

By the Kolmogorov strong law of large numbers for non-identically distributed random variables (Theorem 2.3.10 in Sen and Singer (1994)), we get

$$\frac{1}{n}\sum_{i=1}^n Y_i^2 - \left(\frac{1}{n}\sum_{i=1}^n Y_i\right)^2 - \left[\frac{1}{n}\sum_{i=1}^n E[Y_i^2] - \left(\frac{1}{n}\sum_{i=1}^n E[Y_i]\right)^2\right] \xrightarrow{a.s.} 0$$

as

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n E[Y_i] - E[Y] &= \frac{\gamma}{n}\sum_{i=1}^n \left(\frac{1}{\alpha_i} - \frac{1}{\alpha}\right) + \frac{1}{n}\sum_{i=1}^n (\log\sigma_i - \log\sigma) \\ &= \frac{\gamma}{n}\sum_{i=1}^n \frac{-\varepsilon_i}{(\alpha_i^\star)^2} + \frac{1}{n}\sum_{i=1}^n \frac{\eta_i}{\sigma_i^\star}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^n E[Y_i^2] - E[Y^2] \\ &\quad= \frac{\gamma^2+\pi^2/6}{n}\sum_{i=1}^n \left(\frac{1}{\alpha_i^2} - \frac{1}{\alpha^2}\right) - \frac{2\gamma^2}{n}\sum_{i=1}^n \left(\frac{1}{\alpha_i} - \frac{1}{\alpha}\right) \\ &\qquad+ \frac{2\gamma}{n}\sum_{i=1}^n \left(\left(\frac{1}{\alpha_i} - 1\right)\log\sigma_i - \left(\frac{1}{\alpha} - 1\right)\log\sigma\right) + \frac{1}{n}\sum_{i=1}^n \left((\log\sigma_i)^2 - (\log\sigma)^2\right) \\ &\quad= \frac{\gamma^2+\pi^2/6}{n}\sum_{i=1}^n \frac{-2\varepsilon_i}{\tilde{\alpha_i}^3} - \frac{2\gamma^2}{n}\sum_{i=1}^n \frac{-\varepsilon_i}{(\alpha_i^\star)^2} + \frac{2\gamma}{n}\sum_{i=1}^n \left(\frac{1}{\breve{\sigma}_i}\left(\frac{1}{\breve{\alpha}_i} - 1\right)\eta_i - \frac{\log(\breve{\sigma}_i)}{(\breve{\alpha}_i)^2}\varepsilon_i\right) \\ &\qquad+ \frac{1}{n}\sum_{i=1}^n \frac{2\log\tilde{\sigma}_i}{\tilde{\sigma}_i}\eta_i\end{aligned}$$

where $\tilde{\alpha_i}$, $\alpha_i^\star$, $\breve{\alpha}_i$ (respectively $\tilde{\sigma}_i$, $\sigma_i^\star$, $\breve{\sigma}_i$) have ranged between α and α_i (respectively σ and σ_i).

$$\left|\frac{1}{n}\sum_{i=1}^n E[Y_i] - E[Y]\right| \le \frac{\gamma}{\alpha^2(1-c_\alpha)^2}\frac{1}{n}\sum_{i=1}^n |\varepsilon_i| + \frac{1}{\sigma(1-c_\sigma)}\frac{1}{n}\sum_{i=1}^n |\eta_i|$$

$$\left|\frac{1}{n}\sum_{i=1}^{n}E[Y_i^2]-E[Y^2]\right|$$
$$\leq\left(\frac{2\gamma^2+\pi^2/3}{\alpha^3(1-c_\alpha)^3}+\frac{2\gamma^2}{\alpha^2(1-c_\alpha)^2}+\frac{2\gamma\max(|\log(\sigma-\sigma c_\sigma)|,|\log(\sigma+\sigma c_\sigma)|)}{\alpha^2(1-c_\alpha)^2}\right)\frac{1}{n}\sum_{i=1}^{n}|\varepsilon_i|$$
$$+\left(\frac{2\gamma}{\sigma(1-c_\sigma)}\left(\frac{1}{\alpha(1-c_\alpha)}+1\right)+\frac{2\max(|\log(\sigma-\sigma c_\sigma)|,|\log(\sigma+\sigma c_\sigma)|)}{\sigma(1-c_\sigma)}\right)\frac{1}{n}\sum_{i=1}^{n}|\eta_i|$$

Under the conditions $\frac{1}{n}\sum_{i=1}^{n}|\varepsilon_i|\to 0$ and $\frac{1}{n}\sum_{i=1}^{n}|\eta_i|\to 0$, we have

$$\frac{1}{n}\sum_{i=1}^{n}E[Y_i^2]\xrightarrow[n\to\infty]{}E[Y^2]\quad\text{and}\quad\frac{1}{n}\sum_{i=1}^{n}E[Y_i]\xrightarrow[n\to\infty]{}E[Y].$$

By the continuous mapping theorem with $g(x,y)=\frac{\pi}{\sqrt{\max(6(y-x^2)-\frac{\pi^2}{2},\frac{\pi^2}{4})}}$, we obtain

$$\widehat{\alpha}_n-\alpha\xrightarrow{a.s.}0.$$

Proof of Proposition 4.2

Define the covariance-matrix $\Sigma_{\alpha,\sigma}^{(n)}$ as follows:

$$\Sigma_{\alpha,\sigma}^{(n)}:=\begin{pmatrix}\frac{1}{n}\sum_{i=1}^{n}\mathrm{Var}(Y_i) & \frac{1}{n}\sum_{i=1}^{n}\mathrm{Cov}(Y_i,Y_i^2)\\ \frac{1}{n}\sum_{i=1}^{n}\mathrm{Cov}(Y_i,Y_i^2) & \frac{1}{n}\sum_{i=1}^{n}\mathrm{Var}(Y_i^2)\end{pmatrix}.$$

With the conditions of Proposition 4.1, we have

$$\Sigma_{\alpha,\sigma}^{(n)}\xrightarrow[n\to\infty]{}\Sigma_{\alpha,\sigma}:=\begin{pmatrix}\mathrm{Var}(Y) & \mathrm{Cov}(Y,Y^2)\\ \mathrm{Cov}(Y,Y^2) & \mathrm{Var}(Y^2)\end{pmatrix}.$$

By the central limit theorem for non-identically distributed random variables (Theorem 3.3.9 in Sen and Singer (1994)), as $\sup_k E\left[\left\|\begin{pmatrix}Y_k\\ Y_k^2\end{pmatrix}\right\|^4\right]<\infty$, we have

$$\sqrt{n}\left(\begin{pmatrix}\frac{1}{n}\sum_{i=1}^{n}Y_i\\ \frac{1}{n}\sum_{i=1}^{n}Y_i^2\end{pmatrix}-\begin{pmatrix}\frac{1}{n}\sum_{i=1}^{n}E[Y_i]\\ \frac{1}{n}\sum_{i=1}^{n}E[Y_i^2]\end{pmatrix}\right)\xrightarrow{d}\mathcal{N}(0,\Sigma_{\alpha,\sigma}).$$

Under conditions of Proposition 4.2, we get $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}E[Y_i^2]-E[Y^2]\right)\xrightarrow[n\to\infty]{}0$ and $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}E[Y_i]-E[Y]\right)\xrightarrow[n\to\infty]{}0$, then

$$\sqrt{n}\left(\begin{pmatrix}\frac{1}{n}\sum_{i=1}^{n}Y_i\\ \frac{1}{n}\sum_{i=1}^{n}Y_i^2\end{pmatrix}-\begin{pmatrix}E[Y]\\ E[Y^2]\end{pmatrix}\right)\xrightarrow{d}\mathcal{N}(0,\Sigma_{\alpha,\sigma}).$$

By applying the delta-method with $g(x,y)=\frac{\pi}{\sqrt{\max(6(y-x^2)-\frac{\pi^2}{2},\frac{\pi^2}{4})}}$, we obtain the result.

Proof of Proposition 4.3

By the strong law of large numbers for non-identically distributed random variables (Theorem 2.3.10 in Sen and Singer (1994)), we get

$$\frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2 - \left[\frac{1}{n}\sum_{i=1}^{n} E[Y_i^2] - \left(\frac{1}{n}\sum_{i=1}^{n} E[Y_i]\right)^2\right] \xrightarrow{a.s.} 0.$$

With the same calculation, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} E[Y_i^2] - E[Y^2] &= \frac{\gamma^2+\pi^2/6}{n}\sum_{i=1}^{n} E\left[\frac{1}{\alpha_i^2} - \frac{1}{\alpha^2}\right] + \frac{-2\gamma^2+2\log(\sigma)}{n}\sum_{i=1}^{n} E\left[\frac{1}{\alpha_i} - \frac{1}{\alpha}\right] \\ &= \frac{\gamma^2+\pi^2/6}{n}\sum_{i=1}^{n} E\left[\frac{-2\varepsilon_i}{(\tilde{\alpha_i})^3}\right] + \frac{-2\gamma^2+2\log(\sigma)}{n}\sum_{i=1}^{n} E\left[\frac{-\varepsilon_i}{(\alpha_i^\star)^2}\right]\end{aligned}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} E[Y_i] - E[Y] = \frac{\gamma}{n}\sum_{i=1}^{n} E\left[\frac{1}{\alpha_i} - \frac{1}{\alpha}\right] = \frac{\gamma}{n}\sum_{i=1}^{n} E\left[\frac{-\varepsilon_i}{(\alpha_i^\star)^2}\right]$$

with $\alpha_i^\star$ and $\tilde{\alpha_i} \in (\min(\alpha,\alpha_i), \max(\alpha,\alpha_i))$.

As $\alpha_i^\star$ and $\tilde{\alpha_i}$ are almost surely included in $[\alpha(1-c_\alpha), \alpha(1+c_\alpha)]$, we get

$$\left|\frac{1}{n}\sum_{i=1}^{n} E[Y_i^2] - E[Y^2]\right| \leq \left(\frac{\gamma^2+\pi^2/6}{(\alpha(1-c_\alpha))^3} + \frac{|-2\gamma^2+2\log(\sigma)|}{(\alpha(1-c_\alpha))^2}\right)\frac{1}{n}\sum_{i=1}^{n} E[|\varepsilon_i|]$$

and

$$\left|\frac{1}{n}\sum_{i=1}^{n} E[Y_i] - E[Y]\right| \leq \frac{\gamma}{(\alpha(1-c_\alpha))^2}\frac{1}{n}\sum_{i=1}^{n} E[|\varepsilon_i|].$$

If we suppose $\frac{1}{n}\sum_{i=1}^{n} E[|\varepsilon_i|] \to 0$, we get

$$\frac{1}{n}\sum_{i=1}^{n} E[Y_i^2] \xrightarrow[n\to\infty]{} E[Y^2] \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} E[Y_i] \xrightarrow[n\to\infty]{} E[Y].$$

The central limit theorem is proved as Prop. 4.2.

Acknowledgment

We thank the referees for their valuable comments which helped to improve this article.
J. Lévy Véhel gratefully acknowledges financial support from SMABTP.

Bibliography

Aït-Sahalia, Y. and Jacod, J. (2009). Estimating the degree of activity of jumps in high frequency data. *The Annals of Statistics*, 37.

Belomestny, D. (2010). Spectral estimation of the fractional order of a lévy process. *Ann. Statist.*, 38(1):317–351.

Bergström, H. (1952). On some expansions of stable distribution functions. *Arkiv för Matematik*, 2.

Bianchi, S., Pantanella, A., and Pianese, A. (2013). Modeling stock prices by multifractional brownian motion: an improved estimation of the pointwise regularity. *Quantitative Finance*, 13(8):1317–1330.

Corlay, S., Lebovits, J., and Véhel, J. L. (2014). Multifractional stochastic volatility models. *Mathematical Finance*, 24(2):364–402.

De Haan L., F. A. (2006). *Extreme Value Theory: An Introduction*. 1st edition edition.

Dimitriadis, I. A., Alberola-López, C., Martín-Fernández, M., de-la Higuera, P. C., Simmross-Wattenberg, F., and Asensio-Pérez, J. I. (2011). Anomaly detection in network traffic based on statistical inference and α-stable modeling. *IEEE Transactions on Dependable and Secure Computing*, 8:494–509.

DuMouchel, W. (1983). Estimating the stable index α in order to measure tail thickness: A critique. *Annals of Statistics*, 11.

DuMouchel, W. H. (1973). On the asymptotic normality of the maximum-likelihood estimate when sampling from a stable distribution. *The Annals of Statistics*, 1.

DuMouchel, W. H. (1975). Stable distributions in statistical inference: 2. information from stably distributed samples. *Journal of the American Statistical Association*, 70.

Falconer, K. J. and Lévy Véhel, J. (2009). Multifractional, multistable, and other processes with prescribed local form. *Journal of Theoretical Probability*, 22(2):375–401.

Falconer, K. J. and Lévy Véhel, J. (2018a). Self-stabilizing processes. *Stochastic Models*.

Falconer, K. J. and Lévy Véhel, J. (2018b). Self-stabilizing processes based on random signs. *Journal of Theoretical Probability*.

Fama, E. F. and Roll, R. (1971). Parameter estimates for symmetric stable distributions (ref: V63 p817-36). *Journal of the American Statistical Association*, 66:331–338.

Fischer, R., Akay, M., Castiglioni, P., and Di Rienzo, M. (2003). Multi-and monofractal indices of short-term heart rate variability. *Medical and Biological Engineering and Computing*, 41(5):543–549.

Fofack, H. and Nolan, J. P. (1999). Tail behavior, modes and other characteristics of stable distributions. *Extremes*, 2(1):39–58.

Frezza, M. (2018). A fractal-based approach for modeling stock price variations. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 28(9):091102.

Hall, P. (1982). On some simple estimates of an exponent of regular variation. *Journal of the Royal Statistical Society: Series B (Methodological)*, 44(1):37–42.

Hill, B. M. (1975). A simple general approach to inference about the tail of a distribution. *The Annals of Statistics*, 3.

Jing, B.-Y., Kong, X.-B., Liu, Z., and Mykland, P. (2012). On the jump activity index for semimartingales. *Journal of Econometrics*, 166(2):213 – 223.

Kateregga, M., Mataramvura, S., Taylor, D., and Zhang, X. (2017). Parameter estimation for stable distributions with application to commodity futures log-returns. *Cogent Economics & Finance*, 5.

Koutrouvelis, I. A. (1980). Regression-type estimation of the parameters of stable laws. *J. Amer. Statist. Assoc.*, 75(372):918–928.

Koutrouvelis, I. A. (1981). An iterative procedure for the estimation of the parameters of stable laws. *Comm. Statist. B—Simulation Comput.*, 10(1):17–28.

Kuruoglu, E. E. (2001). Density parameter estimation of skewed alpha-stable distributions. *IEEE Transactions on Signal Processing*, 49(10):2192–2201.

Lavancier, F. and Rochet, P. (2016). A general procedure to combine estimators. *Comput. Statist. Data Anal.*, 94:175–192.

Le Guével, R. (2013). An estimation of the stability and the localisability functions of multistable processes. *Electron. J. Stat.*, 7:1129–1166.

Le Guével, R. and Lévy Véhel, J. (2012). A Ferguson - Klass - LePage series representation of multistable multifractional processes and related processes. *Bernoulli*, 18(4):1099–1127.

Ma, X. and Nikias, C. L. (1995). Parameter estimation and blind channel identification in impulsive signal environments. *IEEE transactions on signal processing*, 43(12):2884–2897.

Mandelbrot, B. B. (1997). *Fractals and Scaling In Finance: Discontinuity, Concentration, Risk*. Springer Publishing Company, Incorporated, 1st edition.

McCulloch, J. (1986). Simple consistent estimators of stable distribution parameters. *Comm. Statist. B—Simulation Comput.*, 15(4):1109–1136.

McCulloch, J. (1997). Measuring tail thickness to estimate the stable index alpha: A critique. *Journal of Business & Economic Statistics*, 15:74–81.

Nolan, J. P. (2001). Maximum likelihood estimation and diagnostics for stable distributions. In *Lévy processes*, pages 379–400. Springer.

Resnick, S. I. (2007). *Heavy-tail phenomena: probabilistic and statistical modeling*. Springer series in operations research and financial engineering. Springer.

Salas-Gonzalez, D., Górriz, J. M., Ramírez, J., Illán, I. A., and Lang, E. W. (2013). Linear intensity normalization of fp-cit spect brain images using the α-stable distribution. *NeuroImage*, 65:449 – 455.

Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable non-Gaussian random processes*. Stochastic Modeling. Chapman & Hall, New York. Stochastic models with infinite variance.

Sen, P. and Singer, J. (1994). *Large Sample Methods in Statistics: An Introduction with Applications*. Chapman & Hall/CRC Texts in Statistical Science. Taylor & Francis.

Stoev, S. and Taqqu, M. (2004). Simulation methods for linear fractional stable motion and farima using the fast fourier transform. *Fractals*, 12(1):95–121.

Wang, X., Li, K., Gao, P., and Meng, S. (2015). Research on parameter estimation methods for alpha stable noise in a laser gyroscope's random error. *Sensors*, 15(8):18550–18564.

Weron, R. (1995). Performance of the estimators of stable law parameters. HSC Research Reports HSC/95/01, Hugo Steinhaus Center, Wroclaw University of Technology.

Yang, C., Hsu, K., and Chen, K. (2009). The use of the levy-stable distribution for geophysical data analysis. *Hydrogeology Journal*, 17(5):1265–1273.